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Output Regulation of Uncertain Nonlinear Systems

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For Cathy and Maria Adelaide

Preface

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories and/or asymptotic rejection of undesired disturbances is a central problem in control theory. A classical setup in which the problem was posed and successfully addressed – in the context of linear, time-invariant and finite dimensional systems – is the one in which the exogenous inputs, namely commands and disturbances, may range over the set of all possible trajectories of a given autonomous linear system, commonly known as the exogeneous system or, more the exosystem.

The case when the exogeneous system is a harmonic oscillator is, of course, classical. Even in this special case, the difference between state and error measurement feedback in the problem of output regulation is profound. To know the initial condition of the exosystem is to know the amplitude and phase of the corresponding sinusoid. On the other hand, to solve the output regulation problem in this case with only error measurement feedback is to track, or attenuate, a sinusoid of known frequency but with unknown amplitude and phase. This is in sharp contrast with alternative approaches, such as exact output tracking, where in lieu of the assumption that a signal is within a class of signals generated by an exogenous system, one instead assumes complete knowledge of the past, present and future time history of the trajectory to be tracked.

The most relevant feature of posing the general problem in these terms is, however, that incorporating a suitable internal model of the exosystem into the compensator which provides the control action, asymptotic tracking with closed loop stability can be achieved, even in the the presence of variations in certain system parameters, on the basis of a relatively restricted amount of information about the controlled plant. The latter, in fact, usually consists in the actual value of the tracking error alone and does not include explicit access to either the actual trajectory to be tracked nor to the disturbance to be rejected.

The importance of asymptotic tracking, disturbance attenuation, and internal stability in their own right underscores the central role which the problem of output regulation has played in the development of classical and modern automatic control. In addition, by varying either the error function or the exosystem, or by setting either the control or the exogenous variables to zero, a variety of interesting problems is obtained, which must also be effectively addressed.

In particular, to solve the output regulation problem one must also be able to four basic problems. The first is to design (robust) stabilizing state feedback laws for nonlinear control systems. The second problem is to determine conditions for the existence of (stable) forced oscillations, in the special case when the exosystem is a harmonic oscillator. Our approach to this problem, for harmonic oscillators as well as for more general exosytems, is geometric and is based on center manifold theory. Because the characterization of a steady-state response is of independent interest, we outline this geometric approach in detail. Of course, one must also be able to design state feedback laws which shape the steady state response to harmonic forcing for general classes of exosystems, as well as for harmonic oscillators. The existence theory for such feedback laws is embodied in what are now known as the regulator equations, and can also be expressed in terms of the transmission zeros, or zero dynamics, of the plant and exosystem. Finally, to solve the problem of output regulation via error measurements one must be able to design dynamic filters, or compensators, which produce a proxy for the plant-exosystem state for feedback laws achieving the prior objectives. In the more classical case where the appropriate system is observable, or detectable, this can be achieved using standard observer design but, especially for problems involving real parametric uncertainty, this may not always be possible. Fortunately, an alternative procedure for the design of dynamic filters can be based on a combination of the internal model principle and the notion of system immersion - an approach which may prove to be of independent interest in robust control.

These are the basic ingredients to both the output regulation problem and the problem of designing output regulation schemes which are robust against real parametric uncertainty or, as we shall refer to it in this book, robust output regulation. Of course, each of these problems, and their synthesis in the solution of problems of output regulation, has a long history. An exhaustive presentation of the theory of output regulation for linear, time-invariant and finite dimensional systems can be found in the works of Davison, Francis and Wonham. In particular, these papers show that a compensator which solves the problem can always be viewed as the interconnection of two components, called the servocompensator and the stabilizing compensator, whose roles are those of generating the control inputs needed to impose the prescribed asymptotic behavior and to stabilize the resulting closed loop system. In this book, we give a detailed presentation of the geometric approach to output regulation in the linear case.

The study of the corresponding design problem for nonlinear, time-invariant and finite dimensional systems, was initiated to the best of our knowledge in the work of Francis, Wonham and Hepburn. These contributions were followed by our earlier work on solving the problem of output regulation of nonlinear systems for neutrally stable exosystems, which led to the formulation of the so-called nonlinear regulator equations, and their existence theory based on zero dynamics. This work has stimulated more extensive research on special classes of exosystems, and on the development of computational approaches to solving the regulator equations by Huang, Rugh, and Krener, as well as the recent development of methods for robust output regulation by Huang, Khalil and ourselves.

In this book, we give a unified treatment of output regulator theory for linear and nonlinear systems and address a number of issues which were left open in the earlier works on the subject. In particular, we describe an approach to structurally stable regulation which unifies and extends a number of prior existing results. We also address the issue of robust regulation, i.e. the issue of achieving regulation in the presence of parameter uncertainties ranging within a prescribed set. We wish to thank several people, agencies and institutions who have supported our work on nonlinear output regulation, stabilization and control. In particular, it is a pleasure to thank the Air Force Office of Scientific Research, the National Science Foundation, the Ministero per la Ricerca Scientifica e Tecnologica, the McDonnell Douglas Corporation, l'Universitá di Roma "La Sapienza", and the Washington University in St. Louis.

Contents

Chapter 1

Introduction

1.1 The basic ingredients of asymptotic output regulation

In this book we consider problems of output regulation for nonlinear systems modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ e &= h(x, w), \end{aligned} \tag{1.1}$$

with state $x \in X \subset \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, regulated output $e \in \mathbb{R}^m$ and exogenous disturbance input $w \in W \subset \mathbb{R}^r$ generated by an exosystem

$$\dot{w} = s(w). \tag{1.2}$$

We assume that $f(x, u, w)$, $h(x, w)$ and $w(w)$ are C^k functions (for some large k) of their arguments and also that $f(0, 0, 0) = 0$, $h(0, 0) = 0$ and $s(0) = 0$.

Generally speaking, a problem of *local* output regulation is to design a feedback controller so as to obtain a closed loop system in which, when $w(t) = 0$, a certain equilibrium is locally asymptotically stable and, when $w(t) \neq 0$ and sufficiently small, the regulated output $e(t)$ asymptotically decays to 0 as $t \to \infty$. The structure of the controller usually depends on the amount of information available for feedback. Throughout this book, we focus our attention to the case in which the information in question only consists, at each time t, in the value $e(t)$ of the error at this time. In other words, we consider controllers modeled by equations of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, e) \\ u &= \theta(\xi) \end{aligned} \tag{1.3}$$

with state $\xi \in \Xi^\circ \subset \mathbb{R}^\nu$, in which $\eta(\xi, e)$ and $\theta(\xi)$ are C^k functions of their arguments, and $\eta(0,0) = 0$, $\theta(0) = 0$. The purpose of output regulation is to obtain a closed loop system in which, from every initial condition in a neighborhood of the equilibrium $(x, \xi, w) = (0,0,0)$, the response of the regulated output asymptotically converges to 0 as time tends to ∞.

It is important to observe that output regulation entails the problems of the asymptotic tracking of a class of reference trajectories, the disturbance attenuation of a class of disturbances, and the requirement that both attenuation and tracking be achieved while maintaining internal stability of the closed-loop system. In this regard, one may think of the exosystem as consisting of two subsystems, one which generates signals to be tracked, and one which generates the disturbances to be attenuated.

The case when the exosystem is a harmonic oscillator is, of course, classical. Even in this special case, the difference between state and error measurement feedback in the problem of output regulation is profound. To know the initial condition of the exosystem is to know the amplitude and phase of the corresponding sinusoid. On the other hand, to solve the output regulation problem in this case with only error measurement feedback is to track, or attenuate, a sinusoid of known frequency but with unknown amplitude and phase. This is in sharp contrast with alternative approaches, such as exact output tracking, where in lieu of the assumption that a signal is within a class of signals generated by an exogenous system, one instead assumes complete knowledge of the past, present and future time history of the trajectory to be tracked.

In this setup, the problem in question can be formally posed in the following terms.

Local Output Regulation. Given a nonlinear system of the form (1.1) with exosystem (1.2) find, if possible, a controller of the form (1.3) such that:

(a) the equilibrium $(x, \xi) = (0, 0)$ of the unforced closed loop system

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi), 0) \\ \dot{\xi} &= \eta(\xi, h(x, 0)) \end{aligned} \tag{1.4}$$

is locally asymptotically stable in the first approximation,

(b) the forced closed loop system

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi), w) \\ \dot{\xi} &= \eta(\xi, h(x, w)) \\ \dot{w} &= s(w) \end{aligned} \tag{1.5}$$

is such that

$$\lim_{t \to \infty} e(t) = 0$$

for each initial condition $(x(0), \xi(0), w(0))$ in a neighborhood of the equilibrium $(0, 0, 0)$. $\triangleleft$

The importance of asymptotic tracking, disturbance attenuation, and internal stability in their own right underscores the central role which the problem of output regulation has played in the development of classical and modern automatic control. In addition, by varying either the error function $h(x, w)$ or the exosystem, or by setting either the control or the exogenous variables to zero, we obtain a variety of interesting problems which must also be effectively addressed. As a preliminary step in our analysis of the problem, we begin by describing the basic ingredients needed for the solution of a problem of asymptotic output regulation. In this respect, we observe that, to solve the output regulation problem we must also be able to:

(i) design locally exponentially stabilizing state feedback laws for nonlinear control systems,

(ii) determine conditions for the existence of (stable) forced oscillations, in the special case when the exosystem is a harmonic oscillator,

(iii) design state feedback laws which shape the steady state response to harmonic forcing for general classes of exosystems, as well as for harmonic oscillators,

(iv) design dynamic filters, or compensators, which produce a proxy for the system-exosystem state for feedback laws achieving the prior objectives.

These are the basic ingredients to both the output regulation problem and, of course, the problem of designing output regulation schemes which are robust against real parametric uncertainty or, as we shall refer to it in this book, robust output regulation. We conclude this section by reviewing the key features and challenges concerning these basic ingredients.

Exponential Stabilization. This first basic ingredient is indeed a problem concerning the linear approximation

$$\dot{x} = Ax + Bu$$

of the nonlinear control system

$$\dot{x} = f(x, u, 0)$$

at the equilibrium $(x, u, w) = (0, 0, 0)$. In this case, if all systems parameters are known, problem (i) can be solved by any of a number of standard approaches, such as infinite horizon, linear-quadratic optimal control or infinite horizon H_∞ control design. Moreover, our calculations in Section 2.3 show that the general solution with error measurement feedback very easily incorporates any given solution to problem (i) into a general law achieving local output regulation. The same calculations extend to the case where certain plant parameters, denoted by μ, are unknown. In this case, one considers the linear approximation

$$\dot{x} = A_\mu x + B_\mu u \tag{1.6}$$

of a nonlinear control system

$$\dot{x} = f_\mu(x, u, 0)\ .$$

The problem of stabilizing the linearization (1.6), for all parametric uncertainties within the class of n-dimensional systems with m-dimensional inputs, with a linear controller has been shown to be NP-hard. On the other hand, if one admits nonlinear controllers and insists that the pair (A_μ, B_μ) be controllable for every μ, then adaptive stabilization schemes are known, underscoring the potential usefulness of nonlinear output regulation, even in a linear context. Needless to say, however, this is a topic which deserves a great deal of further research.

Forced Oscillations and the Existence of a Steady-State Response. For constant coefficient, linear control systems

$$\dot{x} = Ax + Dw, \qquad w(t) = U\sin(\omega t)$$

where no eigenvalue of A lies on the imaginary axis, problem (ii) has a complete and satisfying resolution. Viewing ω as fixed by the choice of the harmonic oscillator with frequency ω as the exosystem, for each amplitude U there is exactly one initial condition x° which generates a periodic trajectory with period $T = 2\pi/\omega$. Moreover, the periodic orbit is asymptotically stable if, and only, if the unforced system is, i.e. whenever the eigenvalues of A all lie in the open left half plane, which we can assume has been arranged as in problem (i).

For nonlinear control systems

$$\dot{x} = f(x, 0, w) \tag{1.7}$$

where, for example, $w(t) = U\sin(\omega t)$, the situation is far more complex, with the possibility of one, or several, forced oscillations with varying stability characteristics occurring. In addition, the fundamental harmonic of these periodic responses may agree with the frequency of the forcing term (harmonic oscillations), or with integer multiples or divisors of the forcing frequency (higher harmonic, or subharmonic, oscillations). Despite a vast literature on nonlinear oscillations,[1] a subject with its origins in celestial mechanics, only for second order systems is there much known about the stability of forced oscillation and, in particular, which of these three kinds of periodic responses might be asymptotically stable.

It is interesting, then, to view the solution of the problem of output regulation for local and more global problems in this context. In this brief introduction we shall limit ourselves to a discussion about the local case (see, however, section 4.5 for a semiglobal analysis). Suppose, for simplicity of discussion, that (1.7) is affine in w, i.e., so that we have

$$\dot{x} = f_0(x) + p(x)w\ ,$$

or, in particular,

$$\dot{x} = f_0(x) + p(x)U\sin(\omega t)$$

and we can view $p(x)U\sin(\omega t)$ as a small perturbation. This is the basis of the method of averaging for determining the existence and

[1] See for example Hayashi [18].

stability of periodic orbits. A similar analysis is motivation for the method of "harmonic balance" in which a Fourier series is considered for an assumed periodic trajectory of period $T = 2\pi/\omega$ and the Fourier coefficients are determined so that the differential equation is satisfied; i.e., so that

$$\int_0^T f(x(s), 0, U\sin(\omega s))ds = 0\,. \tag{1.8}$$

Averaging and harmonic balance have their origin in the method of "small parameters," pioneered by Poincarè. Briefly, in the case of constant coefficients linear control systems, the initial condition x° which generates a periodic trajectory can be expressed as a linear function of the amplitude U

$$x^\circ = PU$$

using the variation of parameters formula, so that

$$P = (I - e^{AT})^{-1}\int_0^T e^{A(T-s)}D\sin(\omega s)ds\,.$$

Poincarè's idea was to take the linear function obtained for the first approximation as the first term in a power series representation in U,

$$x^\circ = PU + P_2U^2 + P_3U^3 + \cdots$$

which can in principle be developed from (1.8) for an assumed periodic trajectory of period T or, alternatively, for an initial condition which generates this trajectory. Of course, one needs to check that the series converges and represents a function $\pi(U)$ in a neighborhood of $U = 0$. Our approach is geometric and seeks instead to "sum the series"

$$x^\circ = PU + P_2U^2 + P_3U^3 + \cdots = \pi(U)$$

by characterizing $\pi(U)$ as the solution of a system of partial differential equations, for which the center manifold theorem guarantees existence of a solution. Moreover, once one has implemented a solution to problem (i), asymptotic (orbital) stability of the steady-state response also follows from the center manifold theorem. Furthermore, for the problem of output regulation treated in this book, the center manifold can be shown to be unique.

Since the existence of a steady-state response to harmonic forcing, or to forcing by more general exosystems, is of independent interest in many applications of systems analysis and control, we shall present this approach in more detail in section 1.2. For a nonlinear control system with unknown parameters

$$\dot{x} = f_\mu(x, u, 0)$$

these arguments persist, yielding an asymptotically stable, steady-state response which, of course, is dependent on μ, and hence unknown for purposes of problem (iii).

Shaping the Steady-State Response. The third basic ingredient in output regulation, the ability to shape the steady-state response of a nonlinear control system

$$\dot{x} = f(x, u, w)$$

is prescriptive in nature, and would appear to be quite challenging from the point of view of the more descriptive tools for predicting the existence of a steady-state response. In our approach to classical and robust output regulation, the existence of a steady-state response and the ability to shape this response are embodied in each of two equations, known as the "regulator equations," which are presented in section 2.1 and whose solution necessarily exists if the problem of local output regulation can be solved using any method.

While the first of the regulator equations will always have a solution, as a consequence of the center manifold theorem, the solvability of the system of regulator equations can be expressed in a system theoretic framework which has an appealing "frequency domain" interpretation. In classical automatic control, there is a simple, intuitive condition for solvability of the the problem of output regulation: no transmission zero of the plant to be controlled should coincide with a natural frequency of the signal to be tracked (or attenuated). For, while a unique steady-state periodic response to harmonic forcing at such a frequency certainly exist for linear, single-input single-output transfer function, if the emitted response is absorbed at this frequency we cannot adjust its amplitude or phase by feedback.

This fundamental condition can be recast in a state-space form by introducing a linear operator whose spectrum on one subspace coincides with the plant transmission zeros and on whose spectrum on another subspace coincides with the natural frequencies of the exosystem. In this setting, to say that no transmission zero of the plant coincides with a natural frequency of the signals to be tracked is to say that these two susbspaces are complementary. In Chapter 3, we describe an existence theory for the regulator equations which provides a nonlinear enhancement of this criterion, in terms of zero dynamics. Briefly, since whenever $w = 0$ and the system output $h(x, 0)$ is constrained to be zero we must have that the error is zero, the zero dynamics of the augmented system contains the zero dynamics of the system to be controlled. In this setting, the regulator equations are solvable just in case the plant zero dynamics are complemented in the augmented zero dynamics by a copy of the exosystem, in a sense we make precise in Chapter 3.

Filtering a Proxy for the Plant/Exosystem State. If the plant/exosystem is exponentially detectable using the error variables as an output measurement, one can verify that a "separation principle" holds for output regulation via state feedback and a standard state observer scheme.[2] Unlike exponential stabilizability of the plant, however, detectability of the augmented system is not a necessary condition for the solvability of the problem of output regulation. More importantly, detectability of the augmented system will not hold when we treat the case of robustness with respect to real, parameteric uncertainty. Rather, the derivation of the regulator equations for the error measurement case reveals that the actual necessary condition relating to measurements of the augmented state is, in fact, a geometric formulation of the "internal model principle." In short, rather than it being necessary to be able to recover the augmented state, it is only necessary that any dynamic compensator which achieves output regulation must also contain a copy of the exosystem.

With these motivations in mind, in section 2.2 we combine the actual necessary conditions, embodied in the regulator equations, with the notion of "immersion" of a nonlinear system to derive a new class of systems which generalize dynamic observers and provide a "proxy" for the state in a compensator design which provides an alternative to the design based on a separation principle. We expect this technique to be of independent interest in other problems of robust control.

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[2] See [34].

1.2 The computation of the steady-state response

Consider a nonlinear system modeled by equations of the form

$$\dot{x} = f(x, u) \tag{1.9}$$

with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}$, where $f(x,u)$ is a C^k function of its arguments with $f(0,0) = 0$, and suppose

$$u(t) = U\sin(\omega t) \ . \tag{1.10}$$

It is easy to see that, if the linear approximation of $\dot{x} = f(x, 0)$ at the equilibrium $x = 0$ is asymptotically stable (i.e. all the eigenvalues of the Jacobian matrix

$$A = \left[\frac{\partial f}{\partial x}\right]_{(0,0)}$$

have negative real part) and U is sufficiently small, then system (1.9) exhibits a well defined steady-state response to the input (1.10). For, observe that the response of system (1.9), in the initial state $x(0) = x_0$ and subject to the input (1.10), coincides with the response of the autonomous system

$$\begin{aligned} \dot{x} &= f(x, d(w)) \\ \dot{w} &= Sw \end{aligned} \tag{1.11}$$

where $w \in \mathbb{R}^2$,

$$S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} , \quad d(w) = w_1 ,$$

and

$$x(0) = x_0 , \quad w(0) = w_0 = \begin{pmatrix} 0 \\ U \end{pmatrix} .$$

If all the eigenvalues of the matrix A have negative real part, system (1.11) has two complementary invariant manifolds through the equilibrium point $(x, w) = (0, 0)$: a stable manifold and a (locally defined) center manifold. The *stable manifold* is the the set of all points $(x, 0)$ such that x belongs to the basin of attraction of the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$. The *center manifold*, on the other hand, can be expressed as the graph of a mapping $x = \pi(w)$, where $\pi(w)$ is a C^{k-1} function satisfying

$$\frac{\partial \pi}{\partial w} S w = f(\pi(w), d(w))$$

and $\pi(0) = 0$.

The restriction of the flow of (1.11) to its center manifold is indeed a copy of the flow of

$$\dot{w} = Sw \, .$$

Thus, since the latter is stable (in the sense of Lyapunov), so is the equilibrium $(x, 0) = (0, 0)$ of the full system (1.11). As a consequence, if U is sufficiently small and $x_0 = \pi(w_0)$, the response $x(t)$ exists for all $t \in \mathbb{R}$ and, in particular, $x(t) = \pi(w(t))$. Set

$$x^{ss}(t) = \pi(w(t)) \, .$$

Since the center manifold is locally attractive, it is readily seen that for every sufficiently small initial condition $x_0 \neq \pi(w_0)$, the response $x(t)$ of (1.11), which exists for all $t \in \mathbb{R}$, differs from $x^{ss}(t)$ but satisfies

$$\lim_{t \to \infty} \|x(t) - x^{ss}(t)\| = 0 \, .$$

Thus, we can conclude that $x^{ss}(t)$ is the (unique) *steady-state response* of (1.9) to the input (1.10).

These considerations can easily be extended to the more general situation in which, instead of harmonic forcing inputs, system (1.9) is driven by inputs generated by an autonomous system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ u &= d(w) \, , \end{aligned} \tag{1.12}$$

where $w \in \mathbb{R}^r$, $s(w)$ and $d(w)$ are C^k functions of their arguments with $s(0) = 0$ and $d(0) = 0$, provided the following hypothesis holds.

Neutral Stability. The equilibrium $w = 0$ is a stable equilibrium (in the sense of Lyapunov) of (1.12) and each initial state $w_0 \in W$ is stable in the sense of Poisson. ◃

An immediate consequence of this hypothesis is that the Jacobian matrix

$$S = \left[\frac{\partial s}{\partial w}\right]_{(0)}$$

has all eigenvalues on the imaginary axis. In fact, no eigenvalue of S can have positive real part, because otherwise the equilibrium $w = 0$ would be unstable. Moreover, since no trajectory of the system can converge to $w = 0$ as $t \to \infty$, no eigenvalue of S can have negative real part. As a consequence of this, if all the eigenvalues of the matrix A have negative real part, the composite system

$$\begin{aligned} \dot{x} &= f(x, d(w)) \\ \dot{w} &= s(w) \end{aligned} \tag{1.13}$$

has a (locally defined) center manifold through $(x, w) = (0, 0)$ and results identical to those discussed above hold.

As an example of this method of determining the steady-state response of a nonlinear system, we describe hereafter how the response in question can be computed for an arbitrary (single-input single-output) finite-dimensional nonlinear system having an input-output map characterized by a *finite Volterra series*, when the forcing input is any finite linear combination of harmonic inputs, i.e. any periodic input having a *finite Fourier series*.

To this end, it indeed suffices to show how to compute the steady-state response of a nonlinear system whose input-output map is characterized by a Volterra series consisting of *one term only*, that is

$$y(t) = \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} w(t, \tau_1, \ldots, \tau_k) u(\tau_1) \ldots u(\tau_k) d\tau_1 \ldots d\tau_k \, . \tag{1.14}$$

Since our method of computing the steady-state response is based on the use of state space models, we first recall an important result about the existence of *finite dimensional* realizations for an input-output map of the form (1.14).

Proposition 1.1 *The following are equivalent*

(i) *the input-output map (1.14) has finite dimensional nonlinear realization,*

(ii) *the input-output map (1.14) has finite dimensional bilinear realization,*

(iii) *there exist matrices* $A_1, A_2, \ldots A_k$, $N_{12}, \ldots, N_{k-1,k}$, C_1 *and* B_k *such that*

$$\begin{aligned} & w(t, \tau_1, \ldots, \tau_k) \\ = \quad & C_1 e^{A_1(t-\tau_1)} N_{12} e^{A_2(\tau_1 - \tau_2)} N_{23} \cdots N_{k-1,k} e^{A_k(\tau_{k-1} - \tau_k)} B_k \, . \end{aligned} \tag{1.15}$$

In particular, from the matrices indicated in condition (iii) it is possible to construct a *bilinear* realization of the map (1.14), which has the form

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + N_{12} x_2 u \\ \dot{x}_2 &= A_2 x_2 + N_{23} x_3 u \\ &\cdots \\ \dot{x}_{k-1} &= A_{k-1} x_{k-1} + N_{k-1,k} x_k u \\ \dot{x}_k &= A_k x_k + B_k u \\ y &= C_1 x_1 \,. \end{aligned} \tag{1.16}$$

In fact, standard calculations show that the Volterra series characterizing the response of (1.16) when its initial state is zero degenerates into a single convolution integral of the form (1.14), in which the kernel $w(t, \tau_1, \ldots, \tau_k)$ has the expression (1.15). The realization in question is possibly non-minimal, but this is not an issue so long as the calculation of the steady-state response is concerned. For convenience, having set

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_{k-1} \\ x_k \end{pmatrix}, \qquad F(x,u) = \begin{pmatrix} A_1 x_1 + N_{12} x_2 u \\ A_2 x_2 + N_{23} x_3 u \\ \cdot \\ A_{k-1} x_{k-1} + N_{k-1,k} x_k u \\ A_k x_k + B_k u \end{pmatrix} \tag{1.17}$$

and $H(x) = C_1 x_1$, system (1.16) will be rewritten as

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \,. \end{aligned}$$

We proceed now with the computation of the steady-state response of this system to an arbitrary (zero mean) input having a finite Fourier series. Indeed, an input of this kind can always be viewed as output of an autonomous linear system of the form

$$\begin{aligned} \dot{w} &= Sw \\ u &= \Gamma w \,, \end{aligned}$$

in which the matrix S is a matrix of the form

$$S = \begin{pmatrix} S_1 & \cdots & 0 \\ \cdot & \cdots & \cdot \\ 0 & \cdots & S_m \end{pmatrix} \tag{1.18}$$

and

$$S_1 = \begin{pmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{pmatrix} \quad \cdots \quad S_m = \begin{pmatrix} 0 & \beta_m \\ -\beta_m & 0 \end{pmatrix} . \tag{1.19}$$

Consistently with the notation used for the blocks of S, we denote the state vector w of this system as

$$w = \mathrm{col}(w_{11}, w_{12}, \dots, w_{m1}, w_{m2}) .$$

In order to be able to use the to the approach outlined at the beginning of the section, we need the following preliminary result.

Lemma 1.2 *Let A be a $n \times n$ matrix having all eigenvalues with nonzero real part and S be as in (1.18) - (1.19). Let $\mathcal{P}$ denote the set of all homogeneous polynomials of degree p in $w_{11}, w_{12}, \dots, w_{m1}, w_{m1}$, with coefficients in $\mathbb{R}$. For any $q(w) \in \mathcal{P}^n$, the equation*

$$\frac{\partial \pi(w)}{\partial w} Sw = A\pi(w) + q(w) \tag{1.20}$$

has a unique solution $\pi(w)$, which is an element of $\mathcal{P}^n$.

Proof. $\mathcal{P}$ is indeed a vector space over $\mathbb{R}$, of finite dimension $d(p,m)$. Set

$$X_i = w_{i1} - jw_{12}, \qquad \bar{X}_i = w_{i1} + jw_{12}$$

and note that any $b(w) \in \mathcal{P}$ can be written as

$$b(w) = \sum_{i_1+j_1+\dots+i_m+j_m=p} b_{i_1 j_1 \cdots i_m j_m} X_1^{i_1} \bar{X}_1^{j_1} \cdots X_m^{i_m} \bar{X}_m^{j_m}$$

where the $b_{i_1 j_1 \cdots i_m j_m}$'s are uniquely determined and

$$b_{i_1 j_1 \cdots i_m j_m} = \bar{b}_{j_1 i_1 \cdots j_m i_m}$$

because the coefficients of $b(w)$ are real numbers. Choose any order for the set of indices $i_1 j_1 \cdots i_m j_m$ and write $b(w)$ in the form

$$b(w) = BW$$

where W is a $d(p,m) \times 1$ vector consisting of all products of the form the $X_1^{i_1} \bar{X}_1^{j_1} \cdots X_m^{i_m} \bar{X}_m^{j_m}$, while B is a $1 \times d(p,m)$ vector consisting of the corresponding $b_{j_1 i_1 \cdots j_m i_m}$'s. In the notation thus established, elements $q(w)$ and $\pi(w)$ of $\mathcal{P}^n$ can be expressed in the form

$$q(w) = QW, \qquad \pi(w) = \Pi W \ ,$$

where Q and Π are $n \times d(p,m)$ matrices.

Note that

$$\frac{\partial (X_1^{i_1} \bar{X}_1^{j_1} \cdots X_m^{i_m} \bar{X}_m^{j_m})}{\partial w} Sw = \lambda_{i_1 j_1 \cdots i_m j_m} X_1^{i_1} \bar{X}_1^{j_1} \cdots X_m^{i_m} \bar{X}_m^{j_m} \ ,$$

where

$$\lambda_{i_1 j_1 \cdots i_m j_m} = j((i_1 - j_1)\beta_1 + \cdots + (i_m - j_m)\beta_m) \ .$$

Thus,

$$\frac{\partial W}{\partial w} Sw = \tilde{S} W$$

where $\tilde{S}$ is a $d(p,m) \times d(p,m)$ diagonal matrix having all the eigenvalues on the imaginary axis.

In the notation introduced above, the equation (1.20) becomes

$$\Pi \tilde{S} W = A \Pi W + QW$$

and this in turn reduces to the Sylvester equation

$$\Pi \tilde{S} = A \Pi + Q \ .$$

Since the spectra of $\tilde{S}$ and A are disjoint, this equation has a unique solution Π. ◁

Using this property it is possible to prove the following result.

Proposition 1.3 *Let $F(x,u)$ be as in (1.17) and S as in (1.18)-(1.19). Assume that all matrices $A_1, A_2, \ldots A_k$ have eigenvalues with negative real part. Then the equation*

$$\frac{\partial \pi(w)}{\partial w} Sw = F(\pi(w), \Gamma w), \qquad \pi(0) = 0 \tag{1.21}$$

has a globally defined solution $\pi(w)$, whose entries are polynomials, in the components of w of degree not exceeding k.

Proof. Set $\pi_k(w) = \Pi_k w$, where Π_k is a matrix of appropriate dimensions. Then observe that the equation

$$\frac{\partial \pi_k(w)}{\partial w} Sw = A_k \pi_k(w) + B_k \Gamma w$$

reduces to a Sylvester equation of the form

$$\Pi_k S = A_k \Pi_k + B_k \Gamma$$

which indeed has a unique solution Π_k because the spectra of S and A_k are disjoint.

Next, consider the equation

$$\frac{\partial \pi_{k-1}(w)}{\partial w} Sw = A_k \pi_{k-1}(w) + N_{k-1,k} \pi_k(w) \Gamma w \,, \tag{1.22}$$

and note that $N_{k-1,k}\pi_k(w)\Gamma w$ is a vector whose entries are homogeneous polynomials of degree 2 in the components of w. Thus, according to Lemma 1.2 this equation has a unique solution $\pi_{k-1}(w)$ whose entries are homogeneous polynomials of degree 2 in the components of w.

By iterating this process, it is easy to show the existence and uniqueness of the solution $\pi(w)$ of (1.21), whose entries are polynomials of degree not exceeding k. ◁

Remark. It is not difficult to extend the previous construction to the case in which the periodically forcing input has nonzero mean value. The details are left to the reader. ◁

The set

$$M_c = \{(x, w) = \pi(w)\} \,,$$

where $\pi(w)$ is the solution of (1.21), is a *globally defined* invariant set for system

$$\begin{aligned} \dot{x} &= F(x, \Gamma w) \\ \dot{w} &= Sw \,. \end{aligned} \tag{1.23}$$

We show next this set is globally attractive and, in particular, that for any pair (x_0, w_0), the solution $x(t)$ of (1.23) satisfying $(x(0), w(0)) = (x_0, w_0)$ and the solution $x^{ss}(t)$ of (1.23) satisfying $(x^{ss}(0), w(0)) = (\pi(w_0), w_0)$ are such that

$$\lim_{t \to \infty} \|x(t) - x^{ss}(t)\| = 0 \,, \tag{1.24}$$

thus concluding that $x^{ss}(t)$ is the steady-state response of (1.23) to the input generated by

$$\begin{aligned} \dot{w} &= Sw \\ u &= \Gamma w \end{aligned}$$

in the initial state $w(0) = w_0$.

First of all we observe that if all matrices $A_1, A_2, \ldots A_k$ have eigenvalues with negative real part, then all trajectories of (1.23) are defined for all t and, in particular, are *bounded*. In fact, for any w_0, the input $u(t) = \Gamma w(t)$ of (1.16) is a bounded function. Since the matrix A_k has all eigenvalues with negative real part, then also the response $x_k(t)$ of the linear system

$$\dot{x}_k = A_k x_k + B_k u$$

from any initial state $x_k(0)$ is bounded. Next consider the subsystem

$$\dot{x}_{k-1} = A_{k-1} x_{k-1} + N_{k-1,k} x_k u$$

viewed as a linear system with input $x_k(t)u(t)$. Since the latter is bounded, then also the response $x_{k-1}(t)$ from any initial state $x_{k-1}(0)$ is bounded. Continuing in the same way, one concludes that the entire response $x(t)$ of

$$\dot{x} = F(x, u)$$

from any initial state $x(0)$ is bounded.

Having established boundedness, the attractivity property (1.24) derives from the following arguments. Set

$$\delta(t) = x(t) - \pi(w(t))$$

and note that, by definition of $\pi(w)$,

$$\dot{\delta}(t) = F(\delta(t) + \pi(w(t)), \Gamma w(t)) - F(\pi(w(t)), \Gamma w(t)) \; .$$

For each fixed u, $F(x, u)$ is linear in x, and can be expressed as

$$F(x, u) = (A + Nu)x$$

with

$$A + Nu = \begin{pmatrix} A_1 & N_{12}u & 0 & \cdots & 0 & 0 \\ 0 & A_2 & N_{23}u & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & A_{k-1} & N_{k-1,k}u \\ 0 & 0 & 0 & \cdots & 0 & A_k \end{pmatrix} . \qquad (1.25)$$

Therefore $\delta(t)$ is a solution of the linear system

$$\dot{\delta}(t) = (A + N\Gamma w(t))\delta(t) .$$

Using the triangularity property of (1.25), the hypothesis that all matrices $A_1, A_2, \ldots, A_k$ have eigenvalues with negative real part and the fact that the components of $w(t)$ are finite combinations of sinusoidal functions, it is easy to deduce that the components $\delta_1(t), \delta_2(t), \ldots,$ $\delta_k(t)$ of $\delta(t)$ can be viewed as responses of cascaded asymptotically stable linear systems. As a consequence

$$\lim_{t\to\infty} \delta(t) = 0 .$$

1.3 Highlights of output regulation for linear systems

For the sake of completeness, we conclude this introductory Chapter with a review of some of the most relevant facts about output regulation of linear systems. This will also facilitate the comparison with a number of results which will be presented later about nonlinear systems.

Consider a linear system modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ e &= Cx + Qw , \end{aligned} \tag{1.26}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, regulated output $e \in \mathbb{R}^m$ and exogenous disturbance input $w \in \mathbb{R}^r$ generated by an exosystem

$$\dot{w} = Sw . \tag{1.27}$$

The problem of output regulation is defined precisely as in section 1.1, that is as the problem of finding a feedback law

$$\begin{aligned} \dot{\xi} &= F\xi + Ge \\ u &= H\xi \end{aligned} \tag{1.28}$$

such that

(a) the equilibrium $(x, \xi) = (0, 0)$ of the unforced closed loop system

$$\begin{aligned} \dot{x} &= Ax + BH\xi \\ \dot{\xi} &= F\xi + GCx \end{aligned} \tag{1.29}$$

is asymptotically stable,

(b) the forced closed loop system

$$\begin{aligned} \dot{x} &= Ax + BH\xi + Pw \\ \dot{\xi} &= F\xi + GCx + GQw \\ \dot{w} &= Sw \end{aligned} \tag{1.30}$$

is such that

$$\lim_{t\to\infty} e(t) = 0$$

for every initial condition $(x(0), \xi(0), w(0))$.

Trivially, if (1.29) is required to be asymptotically stable, i.e the matrix

$$J = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \tag{1.31}$$

is required to have all eigenvalues with negative real part, then (A, B) must be stabilizable and (C, A) must be detectable. On the other hand, asymptotic decay of the error requires a more subtle condition, which reposes on the following simple yet important fact.

Lemma 1.4 *Consider the closed loop system (1.30) and suppose the matrix (1.31) has all eigenvalues with negative real part. Suppose the exosystem is neutrally stable. Then,*

$$\lim_{t\to\infty} e(t) = 0 \tag{1.32}$$

for each initial condition $(x(0), \xi(0), w(0))$ *if and only if there exist matrices* Π *and* Σ *satisfying*

$$\begin{aligned} \Pi S &= A\Pi + BH\Sigma + P \\ \Sigma S &= F\Sigma \\ 0 &= C\Pi + Q\,. \end{aligned} \tag{1.33}$$

Since the result of this Lemma is a particular case of a more general Lemma which will be proven later in section 2.1, we omit its proof and proceed directly with the illustration of a straightforward consequence, which provides a basic necessary condition for the existence of solutions of the problem of output regulation.

Corollary 1.5 *Consider the plant (1.26), with exosystem (1.27). Suppose the exosystem is neutrally stable. There exists a controller* *which solves the problem of output regulation only if there exist matrices* Π *and* Γ *satisfying*

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P \\ 0 &= C\Pi + Q\,. \end{aligned} \tag{1.34}$$

The results expressed by the previous Lemma and by its Corollary are the basic tools also for the analysis of the problem of *robust* output regulation for linear plants subject to parameter uncertainties. Viewing the set of matrices $\{A, B, C, P, Q\}$ which characterize (1.26) as an element of a *space of parameters*

$$\mathcal{P} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times r} \times \mathbb{R}^{m\times r},$$

uncertainty on the values these parameters, within known intervals about certain nominal values, can be simply expressed by allowing the set of parameters $\{A, B, C, P, Q\}$ to range on a given neighborhood $\mathcal{P}_0$ of a fixed element $\{A_0, B_0, C_0, P_0, Q_0\}$ of $\mathcal{P}$. In this setup, the problem of robust regulation for linear plants subject to parameter uncertainties is usually posed as follows.

Robust linear regulator. A fixed controller of the form (1.28) is a robust regulator at $\{A_0, B_0, C_0, P_0, Q_0\}$ if:
(i) it solves the problem of output regulation for the plant characterized by the nominal set of parameters $\{A_0, B_0, C_0, P_0, Q_0\}$,
(ii) it solves the problem of output regulation for each perturbed set of parameters $\{A, B, C, P, Q\}$, so long as the latter is such that the corresponding closed loop system remains asymptotically stable, i.e. is such that the matrix

$$\begin{pmatrix} A & BH \\ GC & F \end{pmatrix}$$

has all eigenvalues with negative real part. ◁

Using the results summarized above and, in particular, Corollary 1.5, it is possible to establish a necessary condition for the existence of a robust regulator. For, suppose a controller of the form (1.28) is a robust regulator at $\{A_0, B_0, C_0, P_0, Q_0\}$ and, consider the set $\mathcal{P}_0$ of all perturbed sets of parameters $\{A, B, C, P, Q\}$ such that the closed loop system remains asymptotically stable. Since, by definition, the matrix

$$\begin{pmatrix} A_0 & B_0H \\ GC_0 & F \end{pmatrix}$$

has all eigenvalues with negative real part, the nominal set of parameters $\{A_0, B_0, C_0, P_0, Q_0\}$ is an interior point of $\mathcal{P}_0$. Again by definition, the fixed regulator (1.28) must solve the problem of output regulation for each perturbed set of parameters $\{A, B, C, P, Q\}$ in $\mathcal{P}_0$. Thus, by Corollary 1.5, the equations (1.34) must have a solution for every $\{A, B, C, P, Q\}$ in $\mathcal{P}_0$ (the solution in question being dependent, of course, on the specific set $\{A, B, C, P, Q\}$). In particular, the equations

$$\begin{aligned} \Pi S &= A_0\Pi + B_0\Gamma + P \\ 0 &= C_0\Pi + Q \end{aligned} \qquad (1.35)$$

must have a solution for every P, Q such that $\{A_0, B_0, C_0, P, Q\}$ in $\mathcal{P}_0$. But since $\{A_0, B_0, C_0, P_0, Q_0\}$ is an interior point of $\mathcal{P}_0$ and (1.35) are linear equations, it follows that the equations in question must have a solution *for all* P, Q.

This observation lends itself to the characterization of a basic *necessary condition* for the existence of a linear robust regulator. To this end, it suffices to recall the following important result about linear matrix equations of the form (1.35).

Proposition 1.6 *The linear equations (1.35) have a solution for all P, Q if and only if the matrix*

$$\begin{pmatrix} A_0 - \lambda I & B_0 \\ C_0 & 0 \end{pmatrix} \qquad (1.36)$$

has independent rows for each λ which is an eigenvalue of S.

Using this result in the present setup and recalling that system (1.26) has the same number of input and output components, it is concluded that a robust regulator for (1.26) exists at $\{A_0, B_0, C_0, P_0, Q_0\}$ *only if* the pair (A_0, B_0) is stabilizable, the pair (C_0, A_0) is detectable, and *the matrix (1.36) is nonsingular for each λ which is an eigenvalue of S.* Fortunately these conditions happen to be also sufficient for the existence of a robust regulator.

Theorem 1.7 *Consider a plant*

$$\begin{aligned} \dot{x} &= A_0x + B_0u + P_0w \\ e &= C_0x + Q_0w, \end{aligned} \qquad (1.37)$$

with exosystem (1.27). Suppose the exosystem is neutrally stable. There exists a robust regulator if and only if the the pair (A_0, B_0) is *stabilizable, the pair (C_0, A_0) is detectable, and the matrix (1.36) is nonsingular for each λ which is an eigenvalue of S.*

To show that the conditions indicated in this Theorem are sufficient, we construct now a controller of the form (1.26) and we prove that this is a robust regulator.

Without loss of generality, suppose that the matrix S which characterizes the exosystem has been transformed into a block-diagonal matrix of the form

$$S = \begin{pmatrix} * & 0 \\ 0 & S_{\min} \end{pmatrix}$$

in which $S_{\min}$ is a matrix whose characteristic polynomial coincides with the minimal polynomial of S (note also that, if this is the case, characteristic polynomial and minimal polynomial of $S_{\min}$ necessarily coincide, i.e. $S_{\min}$ is a *cyclic* matrix). Let q denote the dimension of $S_{\min}$ and let Φ a $qm \times qm$ matrix defined as

$$\Phi = \begin{pmatrix} S_{\min} & 0 & \cdots & 0 \\ 0 & S_{\min} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & S_{\min} \end{pmatrix}.$$

(the block $S_{\min}$ is repeated m times in Φ, where m is the number of input and output components of (1.26)).

Let N and Γ be matrices, of dimensions $qm \times m$ and $m \times qm$ respectively, such that the pair (Φ, N) is controllable and the pair (Γ, Φ) is observable. Standard controllability/observability tests for systems in Jordan form show that matrices of these kind always exist, because Φ, by construction, has exactly m Jordan blocks per each different eigenvalue and the number of columns of N (rows of Γ) is supposed to be equal to m.

At this point, we use the condition that the matrix (1.36) is nonsingular for each λ which is an eigenvalue of S (or, what is the same, eigenvalue of $S_{\min}$ or of Φ), to construct a controller which stabilizes system (1.26) and proves to be a robust regulator. To this end, we need the following result from linear system theory.

Lemma 1.8 *Suppose pairs (A_0, B_0) and (Φ, N) are stabilizable, the pairs (C_0, A_0) and (Γ, Φ) are detectable, and the matrix (1.36) is nonsingular for each λ which is an eigenvalue of Φ. Then, the pair*

$$\begin{pmatrix} A_0 & 0 \\ NC_0 & \Phi \end{pmatrix}, \quad \begin{pmatrix} B_0 \\ 0 \end{pmatrix} \tag{1.38}$$

is stabilizable and the pair

$$\begin{pmatrix} C_0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} A_0 & B_0\Gamma \\ 0 & \Phi \end{pmatrix} \tag{1.39}$$

is detectable.

Proof. Suppose the pair (1.38) is not stabilizable. Then there exist row vectors x^{T} and w^{T} such that

$$\begin{aligned} x^{\mathrm{T}}(A_0 - \lambda I) &= w^{\mathrm{T}} N C_0 \\ x^{\mathrm{T}} B_0 &= 0 \\ w^{\mathrm{T}}(\Phi - \lambda I) &= 0 \end{aligned}$$

for some λ having nonnegative real part. Note that w cannot be zero, otherwise the stabilizability of (A_0, B_0) would be contradicted and therefore λ is necessarily an eigenvalue of Φ (the latter, by the way, in the present situation has all eigenvalues on the imaginary axis). Note also that $u^{\mathrm{T}} = w^{\mathrm{T}} N$ cannot be zero, otherwise the stabilizability of (Φ, N) would be contradicted. Then, the identity

$$\begin{pmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} A_0 - \lambda I & B_0 \\ C_0 & 0 \end{pmatrix}$$

holds, for some nontrivial $(x^{\mathrm{T}}, u^{\mathrm{T}})$, where λ is an eigenvalue of Φ. This contradicts the hypothesis on the matrix (1.36). Identical arguments work to prove detectability of (1.39). ◃

Observe now that, since the pair (1.38) is stabilizable, also the pair

$$\begin{pmatrix} A_0 & B_0\Gamma \\ NC_0 & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B_0 \\ 0 \end{pmatrix} \tag{1.40}$$

is stabilizable (stabilizability is preserved by state feedback) and, since the pair (1.39) is detectable, also the pair

$$\begin{pmatrix} C_0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} A_0 & B_0\Gamma \\ NC_0 & \Phi \end{pmatrix} \tag{1.41}$$

is detectable. As a consequence, the linear system characterized by the triplet of matrices

$$\begin{pmatrix} A_0 & B_0\Gamma \\ NC_0 & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B_0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} C_0 & 0 \end{pmatrix}$$

is stabilizable by output feedback, i.e. there exist K, L, M such that

$$\left(\begin{pmatrix} A_0 & B_0\Gamma \\ NC_0 & \Phi \\ L\,(C_0 & 0) \end{pmatrix} \quad \begin{pmatrix} B_0 \\ 0 \\ K \end{pmatrix} M \right) = \left(\begin{matrix} A_0 \\ \begin{pmatrix} N \\ L \end{pmatrix} C_0 \end{matrix} \quad \begin{matrix} B_0\,(\Gamma \quad M) \\ \begin{pmatrix} \Phi & 0 \\ 0 & K \end{pmatrix} \end{matrix} \right)$$

has all eigenvalues with negative real part.

Using the matrices Φ, N, Γ and K, L, M thus defined, we construct a controller of the form (1.28) with

$$F = \begin{pmatrix} \Phi & 0 \\ 0 & K \end{pmatrix}, \qquad G = \begin{pmatrix} N \\ L \end{pmatrix}, \qquad H = (\Gamma \quad M)\,. \tag{1.42}$$

The controller thus defined does indeed stabilize the nominal plant. In fact, this yields a closed loop system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + B\,(\Gamma \quad M)\,\xi + Pw \\ \dot{\xi} &= \begin{pmatrix} N \\ L \end{pmatrix} Cx + \begin{pmatrix} \Phi & 0 \\ 0 & K \end{pmatrix} \xi + \begin{pmatrix} N \\ L \end{pmatrix} Qw \\ e &= Cx\,, \end{aligned} \tag{1.43}$$

which, if $(A, B, C) = (A_0, B_0, C_0)$, is asymptotically stable by construction.

To check that the proposed controller is a *robust regulator*, we need to check that the error converges asymptotically to zero no matter how the plant parameters are perturbed, so long as the perturbation is such that the corresponding closed loop system remains asymptotically stable. To this end, it is useful to recall the result expressed by Lemma 1.4, which provides a condition under which a controller rendering a closed loop system asymptotically stable also yields asymptotic decay to zero of the associated error. In view of this result, it is clear that we can conclude that the proposed regulator is robust if, *for each set of perturbed plant parameters* (A, B, C, P, Q) for which the closed loop system is asymptotically stable, i.e. such that is the eigenvalues of the matrix

$$\begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \tag{1.44}$$

have negative real part, the equations (1.33) have a solution.

To this end recall that by hypothesis the matrix S which characterizes the exosystem has all eigenvalues on the imaginary axis and therefore, for each set of perturbed plant parameters such that the closed loop system is asymptotically stable, the Sylvester equation

$$\begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} - \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} S = \begin{pmatrix} X \\ Y \end{pmatrix} \tag{1.45}$$

is solvable in Π, Σ for each X, Y on the right-hand side.

Choosing

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -P \\ -GQ \end{pmatrix}$$

we obtain the two equations

$$A\Pi + BH\Sigma + P = \Pi S \tag{1.46}$$

$$F\Sigma - \Sigma S = -G(C\Pi + Q). \tag{1.47}$$

The first condition here coincides with the first condition of (1.33). We need to show that the second condition implies the second and third condition in (1.33). To this end, define two *linear* mappings as follows:

$$\begin{array}{rccc} \mathcal{F}: & \mathbb{R}^{(n+mq)\times r} & \to & \mathbb{R}^{(n+mq)\times r} \\ & \Sigma & \mapsto & \mathcal{F}(\Sigma) = F\Sigma - \Sigma S \\ \mathcal{G}: & \mathbb{R}^{m\times r} & \to & \mathbb{R}^{(n+mq)\times r} \\ & Z & \mapsto & \mathcal{G}(Z) = GZ. \end{array}$$

With this notation, equation (1.47) can be rewritten as

$$\mathcal{F}(\Sigma) + \mathcal{G}(C\Pi + Q) = 0. \tag{1.48}$$

Suppose we are able to prove that the images of $\mathcal{F}$ and $\mathcal{G}$ intersects only at $\{0\}$, so that (1.48) would imply

$$\mathcal{F}(\Sigma) = 0 \tag{1.49}$$

$$\mathcal{G}(C\Pi + Q) = 0. \tag{1.50}$$

In this way we would obtain the second condition of (1.33), which is actually identical to (1.49) in the new notation. Finally, if we were also able to show that $\ker(\mathcal{G}) = \{0\}$, from (1.50) we could deduce that also the remaining condition of (1.33) holds.

In summary, it is possible to conclude that the proof of the fact that the proposed regulator is robust can be completed by simply proving that the two mappings $\mathcal{F}$ and $\mathcal{G}$ satisfy

$$\operatorname{im}(\mathcal{F}) \cap \operatorname{im}(\mathcal{G}) = \{0\} \tag{1.51}$$

$$\ker(\mathcal{G}) = \{0\}. \tag{1.52}$$

The proof of these properties can be achieved via the following auxiliary result.

Lemma 1.9 *There exist at least mr independent solutions Σ of the equation*

$$F\Sigma = \Sigma S. \tag{1.53}$$

Proof. Recall that

$$F = \begin{pmatrix} S_{\min} & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & S_{\min} & 0 \\ 0 & \cdots & 0 & K \end{pmatrix}$$

and observe that if X is a solution of

$$S_{\min} X = XS, \tag{1.54}$$

the m matrices

$$\Sigma_1 = \begin{pmatrix} X \\ 0 \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 \\ X \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \quad \Sigma_m = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ X \\ 0 \end{pmatrix}$$

are independent solutions of (1.53). Since (1.54) has at least r independent solutions X, the result follows. ◁

With the aid of this result, we proceed as follows to prove (1.51) and (1.52). Note that the existence of mr independent solutions of (1.53) shows that the dimension of $\ker(\mathcal{F})$ is at least mr. Since $\mathcal{F}$ maps $\mathbb{R}^{(n+mq)\times r}$ into itself, we deduce that

$$\dim(\mathrm{im}(\mathcal{F})) \le (n+mq)r - mr. \tag{1.55}$$

Moreover,

$$\dim(\mathrm{im}(\mathcal{G})) \le mr \tag{1.56}$$

because the dimension of the image of a linear mapping cannot exceed that of its domain (in this case pr).

Return now to (1.45), pick up any arbitrary X, Y, let $\tilde{\Pi}, \tilde{\Sigma}$ denote the corresponding solution and note that by construction

$$GC\tilde{\Pi} + F\tilde{\Sigma} - \tilde{\Sigma}S = Y$$

i.e.

$$\mathcal{G}(C\tilde{\Pi}) + \mathcal{F}(\tilde{\Sigma}) = Y.$$

Because of the arbitrariness of Y, this relation shows that

$$\mathrm{im}(\mathcal{F}) + \mathrm{im}(\mathcal{G}) = \mathbb{R}^{(n+mq)\times r} \tag{1.57}$$

and this, together with (1.55) and (1.56), yields

$$\begin{aligned} \dim(\mathrm{im}(\mathcal{F})) &= (n+mq)r - mr \\ \dim(\mathrm{im}(\mathcal{G})) &= mr. \end{aligned}$$

These relations prove that (1.51) and (1.52) are true.

Remark. The result of the previous Theorem can be extended to the case of systems having a different number of input and output components. To this end, observe that, using Proposition 1.6, one can still show that a necessary condition for the existence of a robust regulator is that the matrix (1.36) has independent rows for each λ which is an eigenvalue of S. Thus, a robust regulator may exist only if the number of output components does not exceed the number of input components. This being the case, one has to appropriately modify the construction presented above, so as to arrive at a controller which stabilizes the nominal plant and F is such that equation (1.53) has at least pr independent solutions, where p is the number of output components. If this is the case, in fact, the previous arguments still show that the controller in question is a robust regulator. ◁

Chapter 2

Output Regulation of Nonlinear Systems

2.1 The regulator equations

As a preliminary step in our approach to the solution of the problem of local output regulation, for a nonlinear plant modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ e &= h(x, w) \ , \end{aligned} \tag{2.1}$$

we establish a set of elementary *necessary* conditions. First of all, we look at the necessary conditions which derive form the existence of a controller fulfilling the requirement of local internal stability in the first approximation. To this end, let A, B, C, P, Q, S, F, G. H be matrices defined as follows

$$\begin{aligned} A &= \left[\frac{\partial f}{\partial x}\right]_{(0,0,0)} & B &= \left[\frac{\partial f}{\partial u}\right]_{(0,0,0)} & C &= \left[\frac{\partial h}{\partial x}\right]_{(0,0)} \\ P &= \left[\frac{\partial f}{\partial w}\right]_{(0,0,0)} & Q &= \left[\frac{\partial h}{\partial w}\right]_{(0,0)} & S &= \left[\frac{\partial s}{\partial w}\right]_{(0)} \\ F &= \left[\frac{\partial \eta}{\partial \xi}\right]_{(0,0)} & G &= \left[\frac{\partial \eta}{\partial e}\right]_{(0,0)} & H &= \left[\frac{\partial \theta}{\partial \xi}\right]_{(0)} . \end{aligned} \tag{2.2}$$

Then, it is readily seen that the linear approximation of the system

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi), w) \\ \dot{\xi} &= \eta(\xi, h(x, w)) \\ \dot{w} &= s(w) \end{aligned} \tag{2.3}$$

at the equilibrium $(x, u, w) = (0, 0, 0)$, can be expressed in the form

$$\begin{aligned}\dot{x} &= Ax + BH\xi + Pw + \phi(x, \xi, w)\\ \dot{\xi} &= F\xi + GCx + GQw + \chi(x, \xi, w)\\ \dot{w} &= Sw + \psi(w)\end{aligned}$$

where $\phi(x, \xi, w)$, $\chi(x, \xi, w)$ and $\psi(w)$ are functions vanishing at the origin together with their first order derivatives.

In this notation, the requirement of local asymptotic stability in the first approximation is precisely the requirement that the (Jacobian) matrix

$$J = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \tag{2.4}$$

has all eigenvalues with negative real part. Thus, it is immediately concluded that the problem of local output regulation can be solved *only* if the following condition holds.

Linear Stabilizability. The pair (A, B) is stabilizable and the pair (C, A) is detectable. ◃

The condition thus indicated is only related to possibility of fulfilling the requirement local asymptotic stability in the first approximation, and only involves properties of the linear approximation of the plant at the equilibrium $(x, u, w) = (0, 0, 0)$. We now establish another necessary condition, which no longer depends only on the linear approximation of the plant at the equilibrium. The validity of this additional condition reposes on the hypothesis that the exosystem

$$\dot{w} = s(w) \tag{2.5}$$

for which the problem of output regulation is to be solved generates trajectories which are bounded in time, but do not asymptotically decay to 0 as time $t \to \infty$. This hypothesis has been formalized in terms of the concept of ***neutral stability***, introduced earlier in Chapter 1.

Actually, it is easy to prove the following result.

Lemma 2.1 *Consider the closed loop system (2.3). Suppose the exosystem is neutrally stable. Suppose the Jacobian matrix (2.4) has all the eigenvalues with negative real part. Then,*

$$\lim_{t\to\infty} e(t) = 0$$

for each initial condition $(x(0), \xi(0), w(0))$ *in a neighborhood of the equilibrium* $(0,0,0)$ *if and only if there exist mappings* $\pi : W_0 \to \mathbb{R}^n$ *and* $\sigma : W_0 \to \mathbb{R}^\nu$ *(where* $W_0 \subset W$ *is a neighborhood of* $w = 0$*), with* $\pi(0) = 0$ *and* $\sigma(0) = 0$*, such that*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), \theta(\sigma(w)), w) \\ \frac{\partial \sigma}{\partial w} s(w) &= \eta(\sigma(w), 0) \\ 0 &= h(\pi(w), w) \end{aligned} \tag{2.6}$$

for all $w \in W_0$.

Proof. Consider the closed loop system (2.3) and note that the Jacobian matrix of the right-hand side, at the equilibrium $(x, \xi, w) = (0,0,0)$, has the following form

$$\begin{pmatrix} A & BH & * \\ GC & F & * \\ 0 & 0 & S \end{pmatrix} = \begin{pmatrix} J & * \\ 0 & S \end{pmatrix}$$

where J is a matrix with all eigenvalues with negative real part, and S is a matrix with all eigenvalues with zero real part. Thus, the system in question has a center manifold at $(x, w) = (0,0)$, the graph of a mapping $(x, \xi) = (\pi(w), \sigma(w))$ satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \theta(\sigma(w)), w) \tag{2.7}$$

$$\frac{\partial \sigma}{\partial w} s(w) = \eta(\sigma(w), h(\pi(w), w)) . \tag{2.8}$$

Indeed, (2.7) coincides with the first identity in (2.6). To prove that also the second and third identity hold, choose a real number $R > 0$, and let w_0 be a point of W_0, with $\|w_0\| < R$. By hypothesis of neutral stability, the equilibrium $w = 0$ of the exosystem is stable, and it is possible to choose R so that the solution $w(t)$ of (2.5) satisfying $w(0) = w_0$ remains in W_0 for all $t \geq 0$. If $(x(0), \xi(0)) = (x^\circ, \xi_0) = (\pi(w_0), \sigma(w_0))$, the corresponding solution $(x(t), \xi(t))$ of (2.3) will be such that $x(t) = \pi(w(t))$ and $\xi(t) = \sigma(w(t))$ for all $t \geq 0$ because the manifold $(x, \xi) = (\pi(w), \sigma(w))$ is by definition invariant under the flow of (2.3). Since the restriction of the flow of (2.3) to its center manifold is precisely

$$\dot{w} = s(w) ,$$

any point on the center manifold sufficiently close to the origin is Poisson stable by hypothesis. It can be shown that $\lim_{t\to\infty} e(t) = 0$ implies

$$h(\pi(w), w) = 0 \,. \tag{2.9}$$

For, suppose (2.9) is not true at some $(\pi(w_0), w_0)$ sufficiently close to $(0,0)$. Then,

$$M = \|h(\pi(w_0), w_0)\| > 0$$

and there exists a neighborhood V of $(\pi(w_0), \sigma(w_0), w_0)$ such that

$$\|h(\pi(w), w)\| > M/2$$

at each $(\pi(w), \sigma(w), w) \in V$. If $\lim_{t\to\infty} e(t) = 0$ for a trajectory starting at $(\pi(w_0), \sigma(w_0), w_0)$, there exists T such that

$$\|h(\pi(w(t)), w(t))\| < M/2$$

for all $t > T$. But since $(\pi(w_0), \sigma(w_0), w_0)$ is Poisson stable, then for some $t' > T$, $(\pi(w(t')), \sigma(w(t')), w(t')) \in V$ and this contradicts the previous inequality. As a consequence, the identity (2.9), which coincides with the third identity in (2.6), must be true. Finally, (2.9) together with (2.8) yields the second identity in (2.6), and this completes the proof of the necessity.

To prove sufficiency, observe that, because of the third identity in (2.6), the error satisfies

$$e(t) = h(x(t), w(t)) - h(\pi(w(t)), w(t)) \,.$$

As a consequence of the assumptions, the point $(x, \xi, w) = (0, o, 0)$ is a stable equilibrium of (2.3). Then, for sufficiently small $(x(0), \xi(0), w(0))$, the trajectory $(x(t), \xi(t), w(t))$ of (2.3) remains in a small neighborhood of $(0,0,0)$ for all $t \geq 0$. Recall that the center manifold $(x, \xi) = (\pi(w), \sigma(w))$ is locally exponentially attractive, i.e. is such that any trajectory of (2.3) satisfies

$$\|x(t) - \pi(w(t))\| \leq Me^{-at}\|x(0) - \pi(w(0))\|$$

for some $M > 0$ and $a > 0$ and all $t > 0$. Therefore, the continuity of $h(x, w)$ implies

$$\lim_{t\to\infty} e(t) = 0 \,.$$

This completes the proof of the sufficiency. ◁

Remark. It may be convenient to stress that the the result indicated in the previous Lemma is based on the following simple geometric arguments. Observe that, under the hypothesis of neutral stability, if the closed loop system is locally asymptotically stable in the first approximation, then system (2.3) has two complementary invariant manifolds passing through $(x, \xi, w) = (0, 0, 0)$: a stable manifold and a (locally defined) center manifold M_c. The *stable manifold* is the the set of all points $(x, \xi, 0)$ such that (x, ξ) belongs to the basin of attraction of the equilibrium $(x, \xi) = (0, 0)$ of

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi), 0) \\ \dot{\xi} &= \eta(\xi, h(x, 0)) \,. \end{aligned}$$

The *center manifold*, on the other hand, can expressed as the graph of a mapping $w \mapsto (x, \xi) = (\pi(w), \sigma(w))$. Since the center manifold is by definition invariant and locally exponentially attractive, then the property of asymptotic output regulation holds if and only if the error map $h(x, w)$ is zero on M_c (last equation of (2.6)), in which case to say that M_c is invariant for (2.3) is equivalent to say that $\pi(w)$ and $\sigma(w)$ are solutions of the first two equations of (2.6). ◁

Using this result, it is very easy to establish a *necessary* condition for the existence of solutions of a problem of output regulation, which is described in the following statement.

Corollary 2.2 *Consider the plant (2.1), with exosystem (2.5). Suppose the exosystem is neutrally stable. There exists a controller which solves the problem of local output regulation only if there exist mappings $\pi : W_0 \to \mathbb{R}^n$ and $c : W_0 \to \mathbb{R}^m$ (where $W_0 \subset W$ is a neighborhood of $w = 0$), with $\pi(0) = 0$ and $c(0) = 0$, such that*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) \end{aligned} \tag{2.10}$$

for all $w \in W_0$.

Proof. It suffices to set

$$c(w) = \theta(\sigma(w))$$

in the first one of (2.6) to conclude that the mappings $x = \pi(w)$ and $u = c(w)$ necessarily fulfill the identities (2.10). ◁

The set of equations (2.10), which are of paramount importance in the solution of a problem of output regulation, are called the *regulator equations*.

Remark. Note that, in the case of a linear plant

$$\begin{aligned}\dot{x} &= Ax + Bu + Pw\\ e &= Cx + Qu,\end{aligned}$$

with linear exosystem

$$\dot{w} = Sw$$

the regulator equations reduce to the linear matrix equations introduced in section 1.5. In fact, set $\pi(w) = \Pi w$ and $c(w) = \Gamma w$, where Π and Γ are matrices of appropriate dimensions. Then, equations (2.10) reduce to the equations

$$\begin{aligned}\Pi S &= A\Pi + B\Gamma + P\\ 0 &= C\Pi + Q\end{aligned}$$

which have exactly the form (1.34). ◁

So far we have identified two necessary conditions for the solution of the problem of local output regulation: linear stabilizability and existence of a solution for the regulator equations. However, it happens that these conditions are not yet sufficient for the solution of the problem in question. A third condition is needed, which will be described in later the next section. For the moment, we conclude the discussion with an interesting interpretation of the result indicated in the previous corollary. Recall that the first equation in (2.10) expresses the property that the graph of the mapping $x = \pi(w)$ is an *invariant manifold* for the composite system

$$\begin{aligned}\dot{x} &= f(x, c(w), w)\\ \dot{w} &= s(w)\end{aligned} \tag{2.11}$$

while the second one expresses the property that the error map $e = h(x, w)$ is zero at each point of this invariant manifold. Let w^* be any initial state of the exosystem and let

$$w^*(t) = \Phi^s_t(w^*)$$

denote the corresponding exogenous input. If the initial state of the plant is precisely

$$x^* = \pi(w^*)$$

and the input to the plant is precisely equal to

$$u^*(t) = c(w^*(t)) ,$$

it is readily concluded that

$$x(t) = \pi(w^*(t)) ,$$

for all $t \geq 0$ (note that, if w^* is sufficiently small, $w^*(t)$ is defined for all $t \geq 0$ and so is $\pi(w^*(t))$). Thus, since $h(\pi(w), w) = 0$, we have $e(t) = 0$ for all $t \geq 0$. This argument shows that the *control input* generated by the autonomous system

$$\begin{aligned} \dot{w} &= s(w) \\ u &= c(w) \end{aligned} \tag{2.12}$$

from the initial state $w(0) = w^*$ is precisely the input needed to obtain, for the corresponding *exogenous input* $w^*(t)$, a response producing an identically zero error (provided, of course, that the initial condition of the plant is appropriately set, i.e. at $x^* = \pi(w^*)$).

This interpretation leads to the intuition that a controller solving the problem of output regulation must generate a control input consisting of two components: a first component $u^*(t) = c(w^*(t))$ capable of yielding $e(t) = 0$ for all t whenever the initial state of the system is appropriately set (namely, at $x^* = \pi(w^*)$), and a second component capable of rendering the particular trajectory $x^*(t) = \pi(w^*(t))$ locally exponentially attractive.

2.2 The internal model

In the previous section we have shown that, in addition to the trivial condition of linear stabilizability, a necessary condition for the solvability of a problem of output regulation is the existence of two mappings $\pi : W_0 \to \mathbb{R}^n$ and $c : W_0 \to \mathbb{R}^m$ which solve the regulator equations (2.10). As we will see in this section, there is a third condition which needs to be fulfilled for the problem of output regulation to be solvable, which can be expressed as a special property of the system (2.12) (where $c(w)$ is any mapping which renders the regulator equations (2.10) satisfied, for some suitable $\pi(w)$).

In order to conveniently describe this additional condition, a preliminary digression about the notion of *immersion* is in order. Let

$\{X, f, h\}$ denote the autonomous system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x), \end{aligned} \tag{2.13}$$

with state $x \in X$ and output $y \in \mathbb{R}^m$, in which we suppose f to be a smooth vector field and h a smooth mapping, with $f(0) = 0$ and $h(0) = 0$.

System Immersion. System $\{X, f, h\}$ is immersed into system $\{X', f', h'\}$ if there exists a smooth mapping $\tau : X \to X'$, satisfying $\tau(0) = 0$ and

$$h(x) \neq h(z) \Rightarrow h'(\tau(x)) \neq h'(\tau(z)),$$

such that

$$\begin{aligned} \frac{\partial \tau}{\partial x} f(x) &= f'(\tau(x)) \\ h(x) &= h'(\tau(x)) \end{aligned}$$

for all $x \in X$. ◃

Two conditions indicated in this definition express nothing else than the property that any output response generated by $\{X, f, h\}$ is also an output response of $\{X', f', h'\}$. In fact, the first condition implies that the flows $\Phi_t^f(x)$ and $\Phi_t^{f'}(x')$ of the two vector fields f and f' (which are τ-related), satisfy

$$\tau(\Phi_t^f(x)) = \Phi_t^{f'}(\tau(x))$$

for all $x \in X$ and all $t \geq 0$, from which the second condition yields

$$h(\Phi_t^f(x)) = h'(\tau(\Phi_t^f(x))) = h'(\Phi_t^{f'}(\tau(x))),$$

for all $x \in X$ and all $t \geq 0$, thus showing that the output response produced by $\{X, f, h\}$, when its initial state is any $x \in X$, is a response that can also be produced by $\{X', f', h'\}$, if the latter is set in the initial state $\tau(x) \in X'$.

The importance of the notion of immersion in the problem of output regulation depends upon the following crucial result.

Lemma 2.3 *Consider the plant (2.1), with exosystem (2.5). Suppose the exosystem is neutrally stable. There exists a controller which* *solves the problem of local output regulation only if there exist mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W_0 \subset W$ of the origin, satisfying the conditions*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) , \end{aligned}$$

for all $w \in W_0$ and such that the autonomous system with output

$$\begin{aligned} \dot{w} &= s(w) \\ u &= c(w) , \end{aligned}$$

is immersed into a system

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ u &= \gamma(\xi) , \end{aligned}$$

defined on a neighborhood Ξ_0 of the origin in $\mathbb{R}^\nu$, in which $\varphi(0) = 0$ and $\gamma(0) = 0$ and the pair of matrices

$$\Phi = \left[\frac{\partial \varphi}{\partial \xi}\right]_{(0)}, \qquad \Gamma = \left[\frac{\partial \gamma}{\partial \xi}\right]_{(0)} \tag{2.14}$$

is detectable.

Proof. Suppose a controller of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, e) \\ u &= \theta(\xi) \end{aligned} \tag{2.15}$$

solves the problem of output regulation. Then, by Lemma 2.1, there exist mappings $x = \pi(w)$ and $\xi = \sigma(w)$, with $\pi(0) = 0$ and $\sigma(0) = 0$, such that (2.6) are satisfied. Set

$$c(w) = \theta(\sigma(w)), \qquad \gamma(\xi) = \theta(\xi), \qquad \varphi(\xi) = \eta(\xi, 0)$$

and observe that $\pi(w)$ and $c(w)$ satisfy the conditions (2.10) while $\varphi(w)$ and $\gamma(w)$ satisfy

$$\frac{\partial \sigma}{\partial w} s(w) = \varphi(\sigma(w)), \qquad c(w) = \gamma(\sigma(w)) ,$$

thus showing that $\{W_0, s, c\}$ is immersed into $\{\Xi_0, \varphi, \gamma\}$, where $\Xi_0 = \sigma(W_0)$.

Observe now that, by definition, the mappings $\varphi(\xi)$ and $\gamma(\xi)$ introduced above are such that

$$\left[\frac{\partial\eta}{\partial\xi}\right]_{(0,0)} = \left[\frac{\partial\varphi}{\partial\xi}\right]_{(0)}, \qquad \left[\frac{\partial\theta}{\partial\xi}\right]_{(0)} = \left[\frac{\partial\gamma}{\partial\xi}\right]_{(0)}.$$

Since, by hypothesis, the controller (2.15) stabilizes the linear approximation of the plant at the equilibrium $(x, \xi, w) = (0, 0, 0)$, the pair of matrices (2.14) is such that all the eigenvalues of the matrix

$$J = \begin{pmatrix} A & B\Gamma \\ GC & \Phi \end{pmatrix}$$

have negative real part. Since

$$\begin{pmatrix} A & B\Gamma \\ GC & \Phi \end{pmatrix} = \begin{pmatrix} A & 0 \\ GC & \Phi \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \Gamma \end{pmatrix},$$

and

$$\begin{pmatrix} A & B\Gamma \\ GC & \Phi \end{pmatrix} = \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix} + \begin{pmatrix} 0 \\ G \end{pmatrix} \begin{pmatrix} C & 0 \end{pmatrix}$$

it is concluded that the pair

$$\begin{pmatrix} A & 0 \\ GC & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix}$$

is stabilizable and the pair

$$\begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix}, \qquad \begin{pmatrix} C & 0 \end{pmatrix}$$

is detectable. The latter condition implies, in particular, that (Γ, Φ) is detectable. ◁

Thus, a necessary condition for the existence of a solution of the problem of output regulation is that the mapping $c(w)$, which is supposed to satisfy (2.10) for some $\pi(w)$, be such that $\{W_0, s, c\}$ is *immersed into a system* having a *detectable* linear approximation. As we have seen before, a system $\{X, f, h\}$ is immersed into another system $\{X', f', h'\}$ whenever any output response generated by $\{X, f, h\}$ is also an output response of $\{X', f', h'\}$. Thus, a necessary condition for output regulation is that any output response of the system $\{W_0, s, c\}$ is also an output response of some auxiliary system $\{\Xi_0, \varphi, \gamma\}$ having a detectable linear approximation (at the equilibrium $\xi = 0$). As shown at the end of the previous section, for each initial state $w^* \in W_0$, the *output* response of $\{W_0, s, c\}$ is precisely the *control input* needed to yield an error which is identically zero when the plant is driven by the *exogenous input* $w^*(t) = \Phi_t^s(w^*)$. In other words, the result expressed in the previous Lemma says that all the control inputs needed to produce an identically zero error must be generated by some appropriate autonomous dynamical system with output $\{\Xi_0, \varphi, \gamma\}$, having a detectable linear approximation (at the equilibrium $\xi = 0$). This system is called an *internal model* (of the control inputs required to force the error to be identically zero).

We will show in the next section that the existence of an internal model essentially determines the existence of a solution of the problem of output regulation. However, before proceeding with this analysis, in view of its importance in many situations, we discuss hereafter the special case of when a given *nonlinear* system is immersed into a *linear* and observable system. As it is well known, this possibility is best expressed in terms of properties of the so-called *observation space*.

Observation Space The observation space $\mathcal{O}$ of $\{X, f, h\}$ is the smallest subspace of $C^\infty(X)$ (the set of all C^∞ functions defined on X with values in $\mathbb{R}$) which contains $h_1, \ldots, h_m$ and is closed under differentiation along the vector field f. $\triangleleft$

In fact, the following result holds.

Proposition 2.4 *The following are equivalent:*

(i) $\{X, f, h\}$ *is immersed into a finite dimensional and observable linear system,*

(ii) *the observation space* $\mathcal{O}$ *of* $\{X, f, h\}$ *has finite dimension,*

(iii) *there exist an integer* q *and a set of real numbers* $a_0, a_1, \ldots, a_{q-1}$ *such that*

$$L_f^q h(w) = a_0 h(w) + a_1 L_f h(w) + \cdots + a_{q-1} L_f^{q-1} h(w) \,.$$

Proof. To prove that (ii) implies (i) consider, for the sake of simplicity, the case in which $m = 1$ and suppose the observation space $\mathcal{O}$ of $\{X, f, h\}$ has finite dimension r. Then, by definition,

$$h(x), L_f h(x), \ldots, L_f^{r-1} h(x)$$

is a basis of $\mathcal{O}$. In particular the function $L_f^r h(x)$, which is an element of $\mathcal{O}$, can be expressed in the form

$$L_f^r h(x) = a_0 h(x) + a_1 L_f h(x) + \cdots + a_{r-1} L_f^{r-1} h(x)$$

for some set of real numbers a_k, $0 \le k \le r-1$. Thus, $\{X, f, h\}$ is indeed immersed into an observable linear system $\{\mathbb{R}^r, f', h'\}$ in which

$$f'(x') = \begin{pmatrix} x_2' \\ x_3' \\ \cdots \\ x_r' \\ a_0 x_1' + a_1 x_2' + \cdots + a_{r-1} x_r' \end{pmatrix}, \qquad h'(x') = x_1'$$

via

$$\tau(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-2} h(x) \\ L_f^{r-1} h(x) \end{pmatrix}.$$

The extension of these arguments to the case in which $m > 1$ is straightforward.

To prove that (i) implies (iii), observe that by definition

$$\begin{aligned} \frac{\partial \tau}{\partial x} f(x) &= F\tau(x) \\ h(x) &= H\tau(x) , \end{aligned}$$

where F and H are matrices of real numbers. From this it is easy to deduce that

$$L_f^k h(x) = H F^k \tau(x)$$

for any $k \ge 0$. Let

$$d(\lambda) = p_0 + p_1 \lambda + \cdots + p_{q-1} \lambda^{q-1} + \lambda^q$$

denote the minimal polynomial of F. Then

$$p_0 h(x) + p_1 L_f h(x) + \cdots + p_{q-1} L_f^{q-1} h(x) + L_f^q h(x) = H d(F) \tau(x) = 0$$

from which the result follows.

The proof that (iii) implies (ii) is immediate. ◁

2.3 Necessary and sufficient conditions for local output regulation

It is possible now to describe a set of necessary and sufficient conditions for the existence of a solution of the problem of local output regulation. These conditions are centered, as expected, on the existence of a solution of the *regulator equations* (2.10) and can be expressed in terms of the existence of a solution $(\pi(w), c(w))$ satisfying a number of additional requirements.

Theorem 2.5 *Consider the plant (2.1), with exosystem (2.5). Suppose the exosystem is neutrally stable. The problem of local output regulation is solvable if and only if there exist mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W_0 \subset W$ of the origin, satisfying the conditions*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) \,, \end{aligned} \tag{2.16}$$

for all $w \in W_0$ and such that the autonomous system with output

$$\begin{aligned} \dot{w} &= s(w) \\ u &= c(w) \,, \end{aligned}$$

is immersed into a system

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ u &= \gamma(\xi) \,, \end{aligned}$$

defined on a neighborhood Ξ_0 of the origin in $\mathbb{R}^\nu$, in which $\varphi(0) = 0$ and $\gamma(0) = 0$ and the two matrices

$$\Phi = \left[\frac{\partial \varphi}{\partial \xi}\right]_{\xi=0}, \qquad \Gamma = \left[\frac{\partial \gamma}{\partial \xi}\right]_{\xi=0} \tag{2.17}$$

are such that the pair

$$\begin{pmatrix} A & 0 \\ NC & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix} \tag{2.18}$$

is stabilizable for some choice of the matrix N, and the pair

$$\begin{pmatrix} C & 0 \end{pmatrix}, \qquad \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix} \tag{2.19}$$

is detectable.

Proof. The proof of the necessity uses the same arguments as those given in the proof of Lemma 2.3 and needs not to be repeated.

In order to prove sufficiency, choose N so that (2.18) is stabilizable. Then, observe that, as a consequence of the hypotheses on (2.18) and (2.19), also the pair

$$\begin{pmatrix} A & B\Gamma \\ NC & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix}$$

is stabilizable (no matter what Γ is), and the pair

$$(C \quad 0), \qquad \begin{pmatrix} A & B\Gamma \\ NC & \Phi \end{pmatrix}$$

is detectable (no matter what N is). Thus, the linear system characterized by the triplet of matrices

$$F = \begin{pmatrix} A & B\Gamma \\ NC & \Phi \end{pmatrix}, \qquad G = \begin{pmatrix} B \\ 0 \end{pmatrix}, \qquad H = (C \quad 0)$$

is stabilizable by output feedback, i.e. there exist K, L, M so that

$$\begin{pmatrix} \begin{pmatrix} A & B\Gamma \\ NC & \Phi \end{pmatrix} & \begin{pmatrix} B \\ 0 \end{pmatrix} M \\ L(C \quad 0) & K \end{pmatrix}$$

has all eigenvalues with negative real part.

Now, consider the controller

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + Le \\ \dot{\xi}_1 &= \varphi(\xi_1) + Ne \\ u &= M\xi_0 + \gamma(\xi_1) \, . \end{aligned} \tag{2.20}$$

It is easy to see that the controller thus defined solves the problem of output regulation. In fact, it is immediate to see that the Jacobian matrix of the vector field

$$F(x, \xi_0, \xi_1) = \begin{pmatrix} f(x, M\xi_0 + \gamma(\xi_1), 0) \\ K\xi_0 + Lh(x, 0) \\ \varphi(\xi_1) + Nh(x, 0) \end{pmatrix}$$

at $(x, \xi_0, \xi_1) = (0, 0, 0)$, which has the form

$$\begin{pmatrix} A & BM & B\Gamma \\ LC & K & 0 \\ NC & 0 & \Phi \end{pmatrix},$$

has all eigenvalues with negative real part. Moreover, by hypothesis, there exist mappings $x = \pi(w)$, $u = c(w)$ and $\xi_1 = \tau(w)$ such that (2.16) holds and

$$\frac{\partial \tau}{\partial w} s(w) = \varphi(\tau(w)), \qquad c(w) = \gamma(\tau(w)) .$$

This shows that the sufficient conditions of Lemma 2.1 are satisfied by

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \sigma(w) = \begin{pmatrix} 0 \\ \tau(w) \end{pmatrix}$$

and completes the proof of the sufficiency. ◁

The result described by this Theorem shows that there exists a solution of the problem of local nonlinear output regulation if and only if:

(i) the regulator equations have a solution $\{\pi(w), c(w)\}$, with $c(w)$ such that

(ii) there exists an internal model $\{\varphi(\xi), \gamma(\xi)\}$, and the latter is such that

(iii) appropriate stabilizability/detectability conditions hold for the linear approximation of the plant, as well as of the internal model, at the equilibrium point about which regulation is to be achieved.

It is worth observing that standard stabilizability/detectability tests show that the condition that the pair (2.18) is stabilizable *implies* the condition that the pair (A, B) is stabilizable and, similarly, the condition that the pair (2.19) is detectable *implies* the condition that the pair (C, A) is detectable. Thus, the conditions of Theorem 2.5 include – as expected – the trivial necessary conditions of *local stabilizability* requested for the fulfillment of condition local asymptotic stability in the first approximation.

Conversely (see also Lemma 1.8) if the pairs (A, B) and (Φ, N) are stabilizable, the pairs (C, A) and (Γ, Φ) are detectable, and the matrix

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \tag{2.21}$$

is nonsingular for every λ which is an eigenvalue of Φ having nonnegative real part, then the pair

$$\begin{pmatrix} A & 0 \\ NC & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix}$$

is stabilizable and the pair

$$(C \quad 0), \qquad \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix}$$

is detectable.

The proof of the sufficiency of Theorem 2.5 shows how to actually construct a controller which solves the problem of local output regulation. This controller consists of the *parallel connection* of two systems: a system modeled by the equations

$$\begin{aligned} \dot{\xi}_1 &= \varphi(\xi_1) + Ne \\ u &= \gamma(\xi_1)\,, \end{aligned} \tag{2.22}$$

and a system modeled by the equations

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + Le \\ u &= M\xi_0\,. \end{aligned} \tag{2.23}$$

The first subsystem contains an *internal model* of the control inputs required to force the output to be identically zero, and the N is chosen in such a way that the interconnection

$$\begin{aligned} \dot{x} &= f(x, \gamma(\xi_1) + u, w) \\ \dot{\xi}_1 &= \varphi(\xi_1) + Nh(x, w) \\ e &= h(x, w)\,, \end{aligned}$$

is locally asymptotically stabilizable in the first approximation. The role of the second subsystem locally is simply that to render the entire closed loop system locally asymptotically stable in the first approximation.

Observe that the identities

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), \gamma(\tau(w)), w) \\ \frac{\partial \tau}{\partial w} s(w) &= \varphi(\tau(w))\,, \end{aligned}$$

which hold by construction, prove that the submanifold

$$M_c = \{(x, \xi_0, \xi_1, w) : x = \pi(w), \xi_0 = 0, \xi_1 = \tau(w)\}$$

is an invariant manifold of the composite system

$$\begin{aligned} \dot{x} &= f(x, M\xi_0 + \gamma(\xi_1), w) \\ \dot{\xi}_0 &= K\xi_0 + Lh(x, w) \\ \dot{\xi}_1 &= \varphi(\xi_1) + Nh(x, w) \\ \dot{w} &= s(w) \end{aligned} \tag{2.24}$$

which is nothing else than the closed loop system driven by the exosystem. On this manifold the error map $e = h(x, w)$ is zero. This invariant manifold is locally exponentially attractive and therefore, as discussed above, for any sufficiently small initial condition $x(0), \xi_0(0), \xi_1(0), w(0)$, the response of (2.24) is such that the error asymptotically converges to zero as t tends to ∞.

The following Corollary provides a simple set of sufficient conditions for the existence of a solution of the problem of local output regulation.

Corollary 2.6 *Consider the plant (2.1), with exosystem (2.5). Suppose the exosystem is neutrally stable. Suppose the pair (A, B) is stabilizable and the pair (C, A) is detectable. Suppose there exist mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W_0 \subset W$ of the origin, satisfying the conditions (2.16). Suppose also there exists integers $p_1, \ldots, p_m$ and functions*

$$\begin{array}{cccc} \phi_i & : \mathbb{R}^{p_i} & \rightarrow & \mathbb{R} \\ & (\zeta_1, \ldots, \zeta_i) & \mapsto & \phi_i(\zeta_1, \ldots, \zeta_i) \end{array}$$

such that, for all $1 \leq i \leq m$, the i-th component $c_i(w)$ of $c(w)$ satisfies

$$L_s^{p_i} c_i(w) = \phi_i(c_i(w), L_s c_i(w), \ldots, L_s^{p_i - 1} c_i(w)) \,. \tag{2.25}$$

for all $w \in W_0$. Set

$$d_{ij} = \left[\frac{\partial \phi_i}{\partial \zeta_j}\right]_{(0,\ldots,0)}$$

and

$$d_i(\lambda) = d_{i0} + d_{i1}\lambda + \ldots + d_{i,p_i-1}\lambda^{p_i-1} - \lambda^{p_i} \,.$$

Finally, suppose that the matrix

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \tag{2.26}$$

is nonsingular for every λ which is a root of any of the polynomials $d_1(\lambda), \ldots, d_m(\lambda)$ having non-negative real part. Then there exists a controller which solves the problem of local output regulation.

Proof. Condition (2.25) implies that $\{W_0, s, c\}$ is immersed into a system

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ u &= \gamma(\xi) \end{aligned}$$

in which

$$\xi = \begin{pmatrix} \xi_1 \\ \cdots \\ \xi_m \end{pmatrix}, \qquad \xi_i = \begin{pmatrix} \xi_{i1} \\ \cdots \\ \xi_{i,p_i} \end{pmatrix}, \quad i = 1, \ldots, m$$

$$\varphi(\xi) = \begin{pmatrix} \varphi(\xi_1) \\ \cdots \\ \varphi(\xi_m) \end{pmatrix}, \quad \varphi_i(\xi) = \begin{pmatrix} \xi_{i2} \\ \xi_{i3} \\ \cdot \\ \xi_{ip_i} \\ \phi_i(\xi_{i1}, \ldots, \xi_{ip_i}) \end{pmatrix}, \quad i = 1, \ldots, m$$

and

$$\gamma(\xi) = \begin{pmatrix} \gamma(\xi_1) \\ \cdots \\ \gamma(\xi_m) \end{pmatrix}, \quad \gamma_i(\xi) = \xi_{i1}, \quad i = 1, \ldots, m\,.$$

In this case, the eigenvalues of Φ are exactly the roots of the polynomials $d_1(\lambda), \ldots, d_m(\lambda)$. After having chosen a matrix N such that the pair (Φ, N) is stabilizable, for instance

$$N = \mathrm{diag}(N_0, \ldots, N_0)$$

with

$$N_0 = \mathrm{col}(0, 0, \ldots, 0, 1)\,,$$

one can conclude that the remaining conditions of Theorem 2.5 hold since the matrix (2.26) is nonsingular for every λ which is an eigenvalue of Φ. ◁

Finally, we discuss the case of when the problem of regulation is solved by means of a *linear* controller.

Corollary 2.7 *Consider the plant (2.1), with exosystem (2.5). Suppose the exosystem is neutrally stable. Suppose the pair (A, B) is stabilizable and the pair (C, A) is detectable. The problem of local output regulation is solved by a* linear *controller* only if *there exist mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W_0 \subset W$ of the origin, satisfying the conditions (2.16) and such that, for some set of q real numbers $a_0, a_1, \ldots, a_{q-1}$,*

$$L_s^q c(w) = a_0 c(w) + a_1 L_s c(w) + \cdots + a_{q-1} L_s^{q-1} c(w)\,, \qquad (2.27)$$

for all $w \in W_0$.

Conversely, if *these conditions hold* and if *the matrix*

$$\begin{pmatrix} A-\lambda I & B \\ C & 0 \end{pmatrix} \tag{2.28}$$

is nonsingular for every λ which is a root of the polynomial

$$d(\lambda) = a_0 + a_1\lambda + \ldots + a_{q-1}\lambda^{q-1} - \lambda^q$$

having non-negative real part, there exists a linear *controller which solves the problem of local output regulation.*

Proof. Necessity. Condition (2.27) is indeed a necessary condition for the existence of a linear controller

$$\begin{aligned} \dot{\xi} &= F\xi + Ge \\ u &= H\xi \end{aligned}$$

solving the problem of output regulation. In fact, if such a controller exists, from the proof of necessity in Theorem 2.5 it is deduced that

$$\frac{\partial\sigma}{\partial w}s(w) = F\sigma(w), \qquad c(w) = G\sigma(w)$$

for some mapping $\xi = \sigma(w)$. Thus $\{W_0, s, c\}$ is immersed into a linear system and, by Proposition 2.4, condition (2.27) necessarily holds.

Sufficiency. Condition (2.27) implies (see Proposition 2.4) that $\{W_0, s, c\}$ is immersed into a linear observable system. In particular, it is very easy to check that $\{W_0, s, c\}$ is immersed into the linear system

$$\begin{aligned} \dot{\xi} &= \Phi\xi \\ u &= \Gamma\xi \end{aligned}$$

in which

$$\begin{aligned} \Phi &= \mathrm{diag}(\Phi_0, \ldots, \Phi_0) \\ \Gamma &= \mathrm{diag}(\Gamma_0, \ldots, \Gamma_0) \end{aligned}$$

and

$$\Phi_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{q-1} \end{pmatrix}, \qquad \Gamma_0 = (1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0).$$

In this case, the minimal polynomial of Φ is equal to $d(\lambda)$. After having chosen a matrix N such that the pair (Φ, N) is stabilizable, for instance

$$N = \mathrm{diag}(N_0, \ldots, N_0)$$

with

$$N_0 = \mathrm{col}(0, 0, \ldots, 0, 1)\ ,$$

one can conclude that the remaining conditions of Theorem 2.5 hold since the matrix (2.28) is nonsingular for every λ which is an eigenvalue of Φ. $\triangleleft$

2.4 The special case of harmonic exogenous inputs

We discuss in this section the special case in which the class of exogenous inputs against which regulation is to be achieved consists of periodic functions having a finite Fourier series. As seen in the introduction, inputs of this kind are generated by a *linear* exosystem

$$\dot{w} = Sw$$

in which the matrix S has the following form

$$\begin{pmatrix} S_0 & 0 & \cdots & 0 \\ 0 & S_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & S_k \end{pmatrix} \tag{2.29}$$

with $S_0 = 0$ (we consider here the more general case in which the exogenous input possibly have a nonzero mean value), and

$$S_1 = \begin{pmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{pmatrix} \quad \cdots \quad S_m = \begin{pmatrix} 0 & \beta_k \\ -\beta_k & 0 \end{pmatrix}\ .$$

Consistently with the notation used for the blocks of S, we denote the state vector w of this system as

$$w = \mathrm{col}(w_0, w_{11}, w_{12}, \ldots, w_{k1}, w_{k1})\ .$$

Since the exosystem is a linear system, we wish to examine in particular the problem of when of nonlinear regulation can be solved by means of a linear controller. In view of the results established at the end of last section, this requires in particular that the mapping $c(w)$ (which, together with some $\pi(w)$, renders the identities (2.16) satisfied) is such that autonomous system with outputs

$$\dot{w} = Sw$$
$$u = c(w)$$

is immersed into a linear system (or, what is the same, condition (2.27) is fulfilled).

It is not difficult to see that this is the case whenever the mapping $c(w)$ is a *polynomial* in the components $w_{11}, w_{12}, \ldots, w_{k1}, w_{k1}$ of w. For, consider for simplicity the case $m = 1$, let p be a fixed integer and let $\mathcal{P}$ denote the set of all polynomials of degree less than or equal to p in the variables $w_{11}, w_{12}, \ldots, w_{k1}, w_{k1}$ with coefficients in $\mathbb{R}$ and vanishing at $w = 0$. $\mathcal{P}$ is indeed a finite-dimensional vector space over $\mathbb{R}$. If $s(w)$ is linear in w and $c(w) \in \mathcal{P}$, then

$$L_s c(w) = \frac{\partial c}{\partial w} s(w)$$

is still a polynomial in $\mathcal{P}$. Observe that mapping

$$\begin{array}{rccc} D: & \mathcal{P} & \rightarrow & \mathcal{P} \\ & c(w) & \mapsto & \dfrac{\partial c}{\partial w} s(w) \end{array} \tag{2.30}$$

is an $\mathbb{R}$-linear mapping from a finite dimensional vector space to itself and let

$$d(\lambda) = \lambda^q - a_{q-1}\lambda^{q-1} - \cdots - a_1\lambda - a_0$$

denote its *minimal polynomial*. Then

$$D^q - a_{q-1}D^{q-1} - \cdots - a_1 D - a_0 I = 0 .$$

We observe from this property that a relation like (2.27) indeed holds for any polynomial $c(w) \in \mathcal{P}$. Identical arguments apply in case $m > 1$.

Note, however, that the class of all polynomials (of degree not exceeding a fixed number) is only just one particular class of functions for which a condition of the form (2.27) holds. For example, it is easy to check that, in the simple case where the matrix (2.29) above consists of only one block S_1, the class of functions $c(w)$ defined by

$$c(w) = \phi(w_1^2 + w_2^2)p(w) ,$$

in which $\phi(r)$ is any ***arbitrary*** function of the real variable r and $p(w)$ is any polynomial (of degree not exceeding a fixed number), is still such that a condition of the form (2.27) holds. In fact, since

$$\frac{\partial(w_1^2 + w_2^2)}{\partial w} S_1 w = 0 \, ,$$

we have

$$\frac{\partial c}{\partial w} S_1 w = \phi(w_1^2 + w_2^2) \frac{\partial p}{\partial w} S_1 w \, .$$

Therefore, if $p(w) \in \mathcal{P}$,

$$\begin{aligned} & L_s^q c(w) - a_{q-1} L_s^{q-1} c(w) - \cdots - a_1 L_s c(w) - a_0 c(w) \\ & \quad = \phi(w_1^2 + w_2^2)(D^q - a_{q-1} D^{q-1} - \cdots - a_1 D - a_0 I) p(w) = 0 \, . \end{aligned}$$

In the following Theorem, we provide a simple characterization of the set of all mappings $c(w)$ for which a condition of the form (2.27) holds, in the case of a linear exosystem. For convenience, as in the proof of Lemma 1.2, we will express $c(w)$ as a function of the variables

$$X_i = w_{i1} - j w_{i2} \qquad \bar{X}_i = w_{i1} + j w_{i2} \tag{2.31}$$

which are related to the original variables w_{i1}, w_{i2} by an invertible transformation.

Theorem 2.8 *Suppose $c(w)$ is analytic, in a neighborhood of $w = 0$, and $s(w) = Sw$, with S as in (2.29). The following statements are equivalent.*

(i) There is an integer $q > 0$ and a set of real numbers $a_0, a_1, \ldots, a_{q-1}$ such that

$$L_s^q c(w) = a_0 c(w) + a_1 L_s c(w) + \cdots + a_{q-1} L_s^{q-1} c(w) \, ,$$

(ii) There is a finite set $\mathcal{R}$ of nonnegative real numbers such that

$$c(w) = \sum_{|(i_1 - j_1)\beta_1 + \cdots + (i_k - j_k)\beta_k)| \in \mathcal{R}} b(w_0)_{i_1 j_1 \cdots i_k j_k} X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} \, .$$

(iii) There is a finite set $\mathcal{R}$ of nonnegative real numbers such that

$$c(w) = \sum_{|\delta_1 \beta_1 + \cdots + \delta_k \beta_k| \in \mathcal{R}} \phi_{\delta_1 \cdots \delta_k}(w_0, w_{11}^2 + w_{12}^2, \cdots, w_{k1}^2 + w_{k2}^2) \Delta_{\delta_1 \cdots \delta_k}$$

where $\Delta_{\delta_1 \cdots \delta_k} = Y_1^{\delta_1} \cdots Y_k^{\delta_k} \pm \bar{Y}_1^{\delta_1} \cdots \bar{Y}_k^{\delta_k}$ and $Y_i = X_i$ or $\bar{X}_i$.

Proof. Consider, for the sake of simplicity, the case $m = 1$.

"(i) $\Leftrightarrow$ (ii)". Since $c(w)$ is analytic, it can be expanded as

$$c(w) = \sum a_{i_1 j_1 \cdots i_k j_k}(w_0) w_{11}^{i_1} w_{12}^{j_1} \cdots w_{k1}^{i_k} w_{k2}^{j_k} \,.$$

Using (2.31), it is also possible to express $c(w)$ in terms of $X_i, \bar{X}_i$, as

$$c(w) = \sum b_{i_1 j_1 \cdots i_k j_k}(w_0) X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k}$$

where $b_{i_1 j_1 \cdots i_k j_k}(w_0)$ are such that

$$b_{i_1 j_1 \cdots i_k j_k}(w_0) = \bar{b}_{j_1 i_1 \cdots j_k i_k}(w_0)$$

because $c(w)$ is a real-valued function.

It is easy to check that

$$L_s(X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k}) = \lambda_{i_1 j_1 \cdots i_k j_k} X_1^{i_1} \bar{X}_k^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} \,,$$

where

$$\lambda_{i_1 j_1 \cdots i_k j_k} = j((i_1 - j_1)\beta_1 + \cdots + (i_k - j_k)\beta_k) \,,$$

which shows that $X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k}$ is an eigenvector of the linear operator L_s, and $j((i_1 - j_1)\beta_1 + \cdots + (i_k - j_k)\beta_k)$ is the corresponding eigenvalue. Consider any bijection from the set of positive integers $\mathbb{N}$ to the set

$$\Omega = \{X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k}\} \,.$$

This bijection defines a complete order on the set Ω. The function $c(w)$ can be expressed in the following form, using infinite dimensional row vectors and column vectors,

$$c(w) = (\cdots \quad 1 \quad \cdots) \begin{pmatrix} \vdots \\ b_{i_1 j_1 \cdots i_k j_k}(w_0) X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} \\ \vdots \end{pmatrix} .$$

where the column vector is in the order of Ω, however only the terms with nonzero coefficients $b_{i_1 j_1 \cdots i_k j_k}(w_0)$ are listed in it.

Note that

$$\begin{aligned} & L_s b_{i_1 j_1 \cdots i_k j_k}(w_0) X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} \\ & \quad = b_{i_1 j_1 \cdots i_k j_k}(w_0) L_s X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} \\ & \quad = \lambda_{i_1 j_1 \cdots i_k j_k} b_{i_1 j_1 \cdots i_k j_k}(w_0) X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} \,. \end{aligned}$$

Thus

$$
\begin{aligned}
&(c(w) \quad L_s c(w) \quad L_s^2 c(w) \quad \cdots \quad L_s^r c(w))^{\mathrm{T}} \\
&= \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & \lambda_{i_1 j_1 \cdots i_k j_k} & \cdots \\ \cdots & \lambda^2_{i_1 j_1 \cdots i_k j_k} & \cdots \\ & \vdots & \\ \cdots & \lambda^r_{i_1 j_1 \cdots i_k j_k} & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ \\ b_{i_1 j_1 \cdots i_k j_k}(w_0) X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} \\ \\ \vdots \end{pmatrix}
\end{aligned}
$$

Condition (i) can be fulfilled if and only if, for some integer $q > 0$ and real numbers $a_0, \ldots, a_{q-1}$

$$
(a_0 \quad a_1 \quad a_2 \quad \cdots \quad -1) \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & \lambda_{i_1 j_1 \cdots i_k j_k} & \cdots \\ \cdots & \lambda^2_{i_1 j_1 \cdots i_k j_k} & \cdots \\ \cdots & \cdot & \cdots \\ \cdots & \lambda^q_{i_1 j_1 \cdots i_k j_k} & \cdots \end{pmatrix} = 0.
$$

However, since the matrix on the right-hand side is a Vandermonde matrix, a relation of this type can hold, for some finite q, if and only if the set

$$
\{ \lambda_{i_1 j_1 \cdots i_k j_k} : b_{i_1 j_1 \cdots i_k j_k}(w_0) \neq 0 \}
$$

has only finite many different elements, i.e. if the set

$$
\mathcal{R} = \{ |(i_1 - j_1)\beta_1 + \cdots + (i_k - j_k)\beta_k| : b_{i_1 j_1 \cdots i_k j_k}(w_0) \neq 0 \}
$$

has only finite many different elements. This proves that statements (i) and (ii) are equivalent.

"(ii) $\Leftrightarrow$ (iii)". We have already observed that

$$
b_{i_1 j_1 \cdots i_k j_k}(w_0) = \bar{b}_{j_1 i_1 \cdots j_k i_k}(w_0) .
$$

Combining these two terms and factoring out the common factors, we obtain

$$
\begin{aligned}
&b_{i_1 j_1 \cdots i_k j_k}(w_0) X_1^{i_1} \bar{X}_1^{j_1} \cdots X_k^{i_k} \bar{X}_k^{j_k} + \bar{b}_{j_1 i_1 \cdots j_k i_k}(w_0) X_1^{j_1} \bar{X}_1^{i_1} \cdots X_k^{j_k} \bar{X}_k^{i_k} \\
&= (X_1 \bar{X}_1)^{s_1} \cdots (X_k \bar{X}_k)^{s_k} \cdot \\
&\qquad (b_{i_1 j_1 \cdots i_k j_k}(w_0) Y_1^{\delta_1} \cdots Y_k^{\delta_k} + \bar{b}_{i_1 j_1 \cdots i_k j_k}(w_0) \bar{Y}_1^{\delta_1} \cdots \bar{Y}_k^{\delta_k})
\end{aligned}
$$

where $s_h = \min(i_h, j_h)$ for $h = 1, \ldots, k$. The fact that $X_i \bar{X}_i = w_{i1}^2 + w_{i2}^2$ and this relation imply that (ii) and (iii) are equivalent. ◁

2.5 Approximate regulation

In the previous sections, we have seen that a fundamental ingredient in the construction of a controller which solves the problem of output regulation is the *internal model*, i.e. an autonomous system with outputs $\{\Xi_0, \varphi, \gamma\}$ into which $\{W_0, s, c\}$ is immersed. We have also seen that, if the exosystem is a linear system with eigenvalues on the imaginary axis, an internal model (more precisely, a *linear and observable* internal model) can easily be constructed whenever the mapping $c(w)$ is known to satisfy either one of the equivalent conditions (ii), (iii) of the previous Theorem 2.8. In general, however, the construction of an internal model is not an easy task, and this suggests the idea of using an *approximate* internal model (in a sense which will be clarified below), to the purpose of obtaining *approximate* output regulation.

More precisely, it can be shown that, for any integer $p > 0$, it is always possible to find a *linear* internal model (of suitable dimension depending on p) with the property that the corresponding controller yields an error which asymptotically converges to a function $\bar{e}(t)$ satisfying an estimate of the form

$$\|\bar{e}(t)\| \leq E(\|w(t)\|)$$

where $E : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a function such that

$$\lim_{r \to 0^+} \frac{E(r)}{r^p} = 0 \tag{2.32}$$

i.e. is infinitesimal of order higher than p as $r \to 0$.

To this purpose, suppose the integer p is fixed, and consider again the linear mapping D defined by (2.30) in the previous section. Let

$$d(\lambda) = \lambda^q - a_{q-1}\lambda^{q-1} - \cdots - a_1\lambda - a_0$$

denote its *minimal polynomial* and let $\Phi_p \in \mathbb{R}^{q \times q}$ be a matrix (with real entries) having minimal polynomial $d(\lambda)$. Observe, also, that it is always possible to find a vector $N_p \in \mathbb{R}^{q \times 1}$ and a vector $\Gamma_p \in \mathbb{R}^{1 \times q}$ such that the pair (Φ_p, N_p) is controllable and the pair (Γ_p, Φ_p) is observable. Using the triplet (Φ_p, N_p, Γ_p) thus determined, set

$$\Phi = \begin{pmatrix} \Phi_p & 0 & \cdots & 0 \\ 0 & \Phi_p & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Phi_p \end{pmatrix}, \qquad N = \begin{pmatrix} N_p & 0 & \cdots & 0 \\ 0 & N_p & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & N_p \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} \Gamma_p & 0 & \cdots & 0 \\ 0 & \Gamma_p & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & N_p \end{pmatrix}, \tag{2.33}$$

where $\Phi \in \mathbb{R}^{mq \times mq}, N \in \mathbb{R}^{mq \times m}, \Gamma \in \mathbb{R}^{m \times mq}$. The triplet (Φ, N, Γ) defines a system consisting of the aggregate of m identical copies of the linear system characterized by the triplet (Φ_p, N_p, Γ_p).

Consider again a controller having the structure of the controller constructed in the proof of Theorem 2.5, that is a controller of the form

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + Le \\ \dot{\xi}_1 &= \Phi\xi_1 + Ne \\ u &= M\xi_0 + \Gamma\xi_1 \,. \end{aligned} \tag{2.34}$$

Suppose the pair (A, B) is stabilizable, the pair (C, A) is detectable, and the matrix (2.21) is nonsingular for every λ which is a root of the minimal polynomial of D_p. Then it is possible to choose (see section 2.3) K, L, M in (2.34) so that the equilibrium $(x, \xi_0, \xi_1) = (0, 0, 0)$ of the associated closed loop system

$$\begin{aligned} \dot{x} &= f(x, M\xi_0 + \Gamma\xi_1, 0) \\ \dot{\xi}_0 &= K\xi_0 + Lh(x, 0) \\ \dot{\xi}_1 &= \Phi\xi_1 + Nh(x, 0) \,, \end{aligned} \tag{2.35}$$

is locally exponentially stable, i.e. the matrix

$$\begin{pmatrix} A & BM & B\Gamma \\ LC & K & 0 \\ NC & 0 & \Phi \end{pmatrix} \tag{2.36}$$

has all eigenvalues with negative real part.

If this is the case, then each (sufficiently small) exogenous input produces a well-defined steady-state response. In fact, observe that system (2.1), controlled by (2.34) and driven by the exosystem

$$\dot{w} = Sw \,.$$

that is the composite system

$$\begin{aligned}\dot{x} &= f(x, M\xi_0 + \Gamma\xi_1, w)\\ \dot{\xi}_0 &= K\xi_0 + Lh(x, w)\\ \dot{\xi}_1 &= \Phi\xi_1 + Nh(x, w)\\ \dot{w} &= Sw\ ,\end{aligned} \tag{2.37}$$

has a *center manifold* at $(x, \xi_0, \xi_1, w) = (0, 0, 0, 0)$. The latter can be expressed in the form

$$M_c = \{(x, \xi_0, \xi_1, w) : x = \tilde{\pi}(w), \xi_0 = \tilde{\sigma}_0(w), \xi_1 = \tilde{\sigma}_1(w)\}$$

i.e. in the form of the graph of a mapping

$$w \mapsto \begin{pmatrix} \tilde{\pi}(w) \\ \tilde{\sigma}_0(w) \\ \tilde{\sigma}_1(w) \end{pmatrix}$$

where $\tilde{\pi}(w), \tilde{\sigma}_0(w), \tilde{\sigma}_1(w)$ are defined in a neighborhood of $w = 0$ and satisfy the following system of partial differential equations

$$\begin{aligned}\frac{\partial\tilde{\pi}}{\partial w}Sw &= f(\tilde{\pi}(w), M\tilde{\sigma}_0(w) + \Gamma\tilde{\sigma}_1(w), w)\\ \frac{\partial\tilde{\sigma}_0}{\partial w}Sw &= K\tilde{\sigma}_0(w) + Lh(\tilde{\pi}(w), w)\\ \frac{\partial\tilde{\sigma}_1}{\partial w}Sw &= \Phi\tilde{\sigma}_1(w) + Nh(\tilde{\pi}(w), w)\ .\end{aligned} \tag{2.38}$$

The manifold in question is invariant and locally exponentially attractive for the composite system (2.37), which means that - for every initial condition $(x(0), \xi_1(0), \xi_2(0), w(0))$ in a neighborhood of $(0, 0, 0, 0)$ - the response of (2.37) converges, as $t \to \infty$, to a uniquely defined steady-state response, which is determined only by the trajectory $w(t)$ of the exosystem and has the form

$$\begin{aligned}x(t) &= \tilde{\pi}(w(t))\\ \xi_0(t) &= \tilde{\sigma}_0(w(t))\\ \xi_1(t) &= \tilde{\sigma}_1(w(t))\ .\end{aligned}$$

Consequently, the regulated output converges toward a steady-state response of the form

$$\tilde{e}(t) = h(\tilde{\pi}(w(t)), w(t))\ .$$

From this analysis it is deduced that if

$$\|h(\tilde{\pi}(w), w)\| \leq E(\|w\|) \ , \tag{2.39}$$

where $E : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a function satisfying (2.32), then the desired goal of having an error asymptotically converging to a function $\tilde{e}(t)$ satisfying an estimate of the form

$$\|\tilde{e}(t)\| \leq E(\|w(t)\|)$$

is achieved.

As a matter of fact, the controller (2.34) is such that the condition (2.39) holds.

Proposition 2.9 *Consider the plant (2.1) controlled by (2.34). Suppose the exosystem is a neutrally stable linear system. Let Φ, N, Γ be chosen as in (2.33) and let K, L, M be such that the matrix (2.36) has all eigenvalues with negative real part. Then there exists a mapping $\tilde{\pi}(w)$ which satisfies (2.38) for some $\tilde{\sigma}_0(w), \tilde{\sigma}_1(w)$ and is such that*

$$\|h(\tilde{\pi}(w), w)\| \leq E(\|w\|)$$

where $E(w)$ is infinitesimal of order higher than p as $\|w\| \to 0$.

In order to prove Proposition 2.9 we first establish the following Lemma.

Lemma 2.10 *Let $N_p \in \mathbb{R}^{q\times 1}$ be nonzero. Suppose that, for each q-tuple $(\phi_1, \ldots, \phi_q) \in \mathcal{P}^q$ there exists a q-tuple $(\sigma_1, \ldots, \sigma_q) \in \mathcal{P}^q$ and $\gamma \in \mathcal{P}$ satisfying*

$$\begin{pmatrix} \phi_1 \\ \cdots \\ \phi_q \end{pmatrix} = \begin{pmatrix} D\sigma_1 \\ \cdots \\ D\sigma_q \end{pmatrix} - \Phi_p \begin{pmatrix} \sigma_1 \\ \cdots \\ \sigma_q \end{pmatrix} - N_p \gamma.$$

Then, $(\phi_1, \ldots, \phi_q) = (0, \ldots, 0)$ implies $\gamma = 0$.

Proof. Define the linear mappings

$$\begin{array}{cccc} \mathcal{F}: & \mathcal{P}^q & \to & \mathcal{P}^q \\ & \begin{pmatrix} \sigma_1 \\ \cdots \\ \sigma_q \end{pmatrix} & \mapsto & \begin{pmatrix} D\sigma_1 \\ \cdots \\ D\sigma_q \end{pmatrix} - \Phi_p \begin{pmatrix} \sigma_1 \\ \cdots \\ \sigma_q \end{pmatrix} \end{array}$$

and

$$\begin{array}{rcl} \mathcal{G}: \mathcal{P} & \to & \mathcal{P}^q \\ \gamma & \mapsto & N_p\gamma \,. \end{array}$$

The hypothesis of the Lemma is that

$$\operatorname{im}(\mathcal{F}) + \operatorname{im}(\mathcal{G}) = \mathcal{P}^n \,. \tag{2.40}$$

It is easy to check that $\mathcal{F}(\sigma_1, \ldots, \sigma_q) = 0$ has at least ρ independent solutions, where $\rho = \dim(\mathcal{P})$ (see Lemma 1.9). Thus, $\dim(\operatorname{im}(\mathcal{F})) \leq q\rho - \rho$. Moreover, $\dim(\operatorname{im}(\mathcal{G})) \leq \rho$. This, together with (2.40) shows that $\operatorname{im}(\mathcal{F}) \cap \operatorname{im}(\mathcal{G}) = \{0\}$ and the result follows. ◁

Using this result it is easy to arrive at the desired conclusion.

Proof. (of Proposition 2.9). Observe that by hypothesis, the control law (2.34) locally exponentially stabilizes system (2.35). Thus, for *any* function $\lambda(w)$, the system

$$\begin{aligned} \dot{x} &= f(x, M\xi_0 + \Gamma\xi_1, w) \\ \dot{\xi}_0 &= K\xi_0 + Lh(x, w) \\ \dot{\xi}_1 &= \Phi\xi_1 + Nh(x, w) + \lambda(w) \\ \dot{w} &= Sw \end{aligned}$$

has a center manifold at $(x, \xi_0, \xi_1, w) = (0, 0, 0, 0)$. In other words, for any arbitrary vector $\lambda(w)$, there exist $\zeta(w), \eta_0(w), \eta_1(w)$ such that,

$$\begin{aligned} \frac{\partial \zeta}{\partial w} Sw &= f(\zeta(w), M\eta_0(w) + \Gamma\eta_1(w), w) \\ \frac{\partial \eta_0}{\partial w} Sw &= K\eta_0(w) + Lh(\zeta(w), w) \\ \frac{\partial \eta_1}{\partial w} Sw &= \Phi\eta_1(w) + Nh(\zeta(w), w) + \lambda(w) \,. \end{aligned} \tag{2.41}$$

In view of the particular structure of Φ and N, the third of these equations can be written, after having split $\eta_1(w)$ and $\lambda(w)$ as

$$\eta_1(w) = \begin{pmatrix} \eta_{11}(w) \\ \vdots \\ \eta_{1m}(w) \end{pmatrix}, \qquad \lambda(w) = \begin{pmatrix} \lambda_{11}(w) \\ \vdots \\ \lambda_{1m}(w) \end{pmatrix},$$

in the form

$$\frac{\partial \eta_{1i}}{\partial w} Sw = \Phi_p \eta_{1i}(w) + N_p h_i(\zeta(w), w) + \lambda_i(w) \,, \tag{2.42}$$

where $h_i(\zeta(w), w)$ is the i-th component of $h(\zeta(w), w)$.

Suppose now the entries of the (arbitrary) vector $\lambda_i(w)$ are in $\mathcal{P}$ and (uniquely) decompose $\eta_{1i}(w)$ and $h_i(\zeta(w), w)$ as

$$\begin{aligned} \eta_{1i}(w) &= \eta_{1i}^1(w) + \eta_{1i}^2(w) \\ h_i(\zeta(w), w) &= \gamma_i^1(w) + \gamma_i^2(w) \end{aligned}$$

with all entries of $\eta_{1i}^1(w)$ and $\gamma_i^1(w)$ in $\mathcal{P}$ and

$$\lim_{\|w\|\to 0} \frac{\|\eta_{1i}^2(w)\|}{\|w\|^p} = \lim_{\|w\|\to 0} \frac{\|\gamma_i^2(w)\|}{\|w\|^p} = 0 \, .$$

Clearly (2.42) yields

$$\frac{\partial \eta_{1i}^1}{\partial w} S w = \Phi_p \eta_{1i}^1(w) + N_p \gamma_i^1(w) + \lambda_i(w) \, .$$

Since $\lambda_i(w)$ is arbitrary, it is concluded that Φ_p and N_p are such that the hypothesis of Lemma 2.10 is fulfilled. Thus, this relation precisely shows that for each q-tuple $(\phi_1, \ldots, \phi_q) \in \mathcal{P}^q$ there exists a q-tuple $(\sigma_1, \ldots, \sigma_q) \in \mathcal{P}^q$ and $\gamma \in \mathcal{P}$ satisfying

$$\begin{pmatrix} \phi_1 \\ \cdots \\ \phi_q \end{pmatrix} = \begin{pmatrix} D\sigma_1 \\ \cdots \\ D\sigma_q \end{pmatrix} - \Phi_p \begin{pmatrix} \sigma_1 \\ \cdots \\ \sigma_q \end{pmatrix} - N_p \gamma .$$

Thus, using Lemma 2.10 we can conclude that if $\lambda_i(w) = 0$, then $\gamma_i^1(w) = 0$.

Now, observe that equations (2.38) and equations (2.41) are identical, if $\lambda(w) = 0$ (with the obvious replacement of $\tilde{\pi}(w)$, $\tilde{\sigma}_0(w)$, $\tilde{\sigma}_1(w)$ by $\zeta(w)$, $\eta_0(w)$, $\eta_1(w)$). Thus, from the previous arguments we deduce that the solution $\tilde{\pi}(w)$, $\tilde{\sigma}_0(w)$, $\tilde{\sigma}_1(w)$ of (2.38) is such that, if $h(\tilde{\pi}(w), w)$ is decomposed as

$$h(\tilde{\pi}(w), w) = \gamma^1(w) + \gamma^2(w)$$

where the entries of $\gamma^1(w)$ are in $\mathcal{P}$ and

$$\lim_{\|w\|\to 0} \frac{\|\gamma^2(w)\|}{\|w\|^p} = 0$$

then necessarily $\gamma^1(w) = 0$.

In other words

$$\lim_{\|w\|\to 0} \frac{\|h(\tilde{\pi}(w), w)\|}{\|w\|^p} = 0$$

and this concludes the proof of the Proposition. ◃

Chapter 3

Existence Conditions for Regulator Equations

3.1 Linear regulator equations and transmission zeros

The purpose of this Chapter is to show that the existence of solutions for the pair of equations

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) , \end{aligned} \tag{3.1}$$

which, as we have seen in the previous Chapter, determine the existence of solutions of the problem of local output regulation, is intimately related to the properties of the so-called *zero dynamics* of the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ \dot{w} &= s(w) \\ e &= h(x, w) . \end{aligned} \tag{3.2}$$

As we will explain later in the next section, the zero dynamics of a given nonlinear system is essentially the collection of all the (forced) state trajectories which are compatible with the constraint that the output is identically zero for all times. In the case of a linear system, the qualitative behavior of the these trajectories is determined by the zeros of the transfer function of the system itself (for a single-input single-output system) or by the more general notion of *transmission zeros* (for multi-input multi-output systems).

It is well known that, in a linear system, the notion of transmission zeros bears an important relation with the existence of solutions the linear version of the regulator equations. The latter, as seen in section 1.5, in the case of a controlled plant modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ \dot{w} &= Sw \\ e &= Cx + Qw \, , \end{aligned} \tag{3.3}$$

reduce to the pair of linear matrix equations

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P \\ 0 &= C\Pi + Q \, . \end{aligned} \tag{3.4}$$

For the sake of completeness, we review in this section the notion of transmission zeros and the relation between this notion and the existence of solutions of (3.4). Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.5}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $e \in \mathbb{R}^m$ (note that the same number of input and output components is assumed). Suppose the determinant of the matrix

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \tag{3.6}$$

is not identically zero (note that the latter condition is equivalent to the condition that the transfer function matrix of the system, namely $T(s) = C(sI - A)^{-1}B$, is *invertible*). A *transmission zero* of (3.5) is any root of the polynomial

$$n(\lambda) = \det \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$$

satisfying

$$\operatorname{rank} \begin{pmatrix} A - \lambda I & B \end{pmatrix} = n, \quad \operatorname{rank} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n \, .$$

Using the terminology of transmission zeros, the existence of solutions of the linear regulator equations for all P, Q can be expressed in the following terms (see Proposition 1.6).

Proposition 3.1 *Suppose (A, B) is stabilizable and (C, A) is detectable. Suppose all eigenvalues of S have nonnegative real part. The regulator equations (3.4) have a solution for all P, Q if and only if no eigenvalue of S is a transmission zero of (3.5).*

The condition thus derived is concerned with the existence of solutions of the regulator equation *for all* P, Q. If solutions for some *specific* pair P, Q are sought, then weaker conditions are to be expected. In order to discuss this point, it is convenient to recall some important geometric concepts related to the notion of transmission zeros. Consider again system (3.5), and let V^* denote the largest controlled invariant subspace contained in $\ker(C)$, i.e. the largest subspace V^* satisfying $V^* \subset \ker(C)$ and such that, for some matrix F,

$$(A + BF)V^* \subset V^* .$$

Note that since V^* is invariant under $(A + BF)$, then the restriction

$$(A + BF)|_{V^*} : V^* \to V^*$$

is a well-defined linear mapping.

The relation between matrix (3.6) and the objects thus introduced is described in the following statement.

Proposition 3.2 *The determinant of the matrix (3.6) is not identically zero if and only if $\ker(B) = \{0\}$ and $V^* \cap \operatorname{im}(B) = \{0\}$. If this is the case, then the linear mapping $(A + BF)|_{V^*}$ is independent of F and the invariant polynomials of $(A + BF)|_{V^*} - \lambda I$ coincide with the invariant polynomials of (3.6).*

Proof. The complete proof of this result can be found in the literature. To the purpose of the present discussion, i.e. to show the relation between solvability of regulator equations and the objects thus introduced, observe that, if $V^* \cap \operatorname{im}(B) = \{0\}$, it is possible to choose coordinates in such a way that, having partitioned x as $x = \operatorname{col}(x_1, x_2)$, the subspace V^* is described as

$$V^* = \{(x_1, x_2) \in \mathbb{R}^n : x_1 = 0\}$$

and

$$\operatorname{Im}(B) \subset \{(x_1, x_2) \in \mathbb{R}^n : x_2 = 0\} .$$

Since $V^* \subset \ker(C)$, in these coordinates the matrices A, B, C can be split as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} C_1 & 0 \end{pmatrix} .$$

This form is particularly interesting because it can be shown that the matrix A_{22} is actually a matrix representation of the mapping $(A+BF)|_{V^*}$.

In fact, since $AV^* \subset V^* + \mathrm{Im}(B)$, the matrices A_{12} and B_1 satisfy the condition

$$\mathrm{Im}(A_{12}) \subset \mathrm{Im}(B_1) .$$

Therefore there exists a (unique, because B_1 has rank m) matrix F_2 such that

$$B_1 F_2 = -A_{12} .$$

Setting

$$F = \begin{pmatrix} 0 & F_2 \end{pmatrix} \tag{3.7}$$

it follows that

$$A + BF = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} .$$

We see from this that V^* is invariant under $(A + BF)$ and, in particular, that A_{22} is a representation of the linear mapping $(A + BF)|_{V^*}$. It is also easy to deduce that the invariant polynomials of the matrix (3.6) coincide with the invariant polynomials of $A_{22} - \lambda I$. In fact, observing that the invariant polynomials of (3.6) coincide with those of the matrix

$$\begin{pmatrix} A + BF - \lambda I & B \\ C & 0 \end{pmatrix}$$

for any choice of F, choose in the latter F as in (3.7) and take a permutation of rows and columns, to obtain a matrix of the form

$$\begin{pmatrix} \begin{pmatrix} A_{11} - \lambda I & B_1 \\ C_1 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} A_{21} & 0 \end{pmatrix} & A_{22} - \lambda I \end{pmatrix} . \tag{3.8}$$

Using the maximality of V^*, one can prove that the determinant of the matrix

$$\begin{pmatrix} A_{11} - \lambda I & B_1 \\ C_1 & 0 \end{pmatrix} \tag{3.9}$$

is independent of λ (so that its inverse is a polynomial matrix). Thus, the invariant polynomials of (3.8), i.e. those of the matrix (3.6), coincide with those of the matrix $A_{22} - \lambda I$. ◁

We use now the construction outlined in the previous proof to show that the solvability of the regulator equations (3.4) can be given a characterization in geometric terms. Consider again system (3.5) and suppose $\ker(B) = 0$ and $V^* \cap \text{im}(B) = \{0\}$. Rewrite the controlled plant (3.3) in a similar form, that is

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e u_e \\ e &= C_e x_e , \end{aligned} \tag{3.10}$$

where

$$x_e = \begin{pmatrix} x \\ w \end{pmatrix}, \quad A_e = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix}, \quad B_e = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad C_e = \begin{pmatrix} C & Q \end{pmatrix},$$

and let V_e^* denote the largest controlled invariant subspace of (3.10) contained in $\ker(C_e)$.

Simple interchanges of rows and columns show that the matrix

$$\begin{pmatrix} A_e - \lambda I & B_e \\ C_e & 0 \end{pmatrix} = \begin{pmatrix} A - \lambda I & P & B \\ 0 & S - \lambda I & 0 \\ C & Q & 0 \end{pmatrix} \tag{3.11}$$

has a determinant which is not identically vanishing, so that $\ker(B_e) = \{0\}$, $V_e^* \cap \text{im}(B_e) = \{0\}$ and the constructions indicated in the proof of Proposition 3.2 can be repeated. In particular, choose again for A, B and C the forms indicated in that proof, take a corresponding partition of P as

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix},$$

and observe that the invariant polynomials of (3.11), which coincide with those of

$$\begin{pmatrix} A + BF - \lambda I & P & B \\ 0 & S - \lambda I & 0 \\ C & Q & 0 \end{pmatrix} \tag{3.12}$$

for any F (thus, in particular, for any F which renders V^* invariant under $(A + BF)$), coincide with those of the matrix

$$\begin{pmatrix} \begin{pmatrix} A_{11} - \lambda I & B_1 \\ C_1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & P_1 \\ 0 & Q \end{pmatrix} \\ \begin{pmatrix} A_{12} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} A_{22} - \lambda I & P_2 \\ 0 & S - \lambda I \end{pmatrix} \end{pmatrix}.$$

Since, as observed before, the matrix in the upper-left block has a determinant which is independent of λ, this shows that the invariant polynomials of the matrix (3.11), which coincide with the invariant polynomials of $(A_e + B_eF_e)|_{V_e^*} - \lambda I$ (where F_e is any matrix rendering V_e^* invariant under $(A_e + B_eF_e)$), are precisely the invariant polynomials of the matrix

$$\begin{pmatrix} A_{22} - \lambda I & P_2 \\ 0 & S - \lambda I \end{pmatrix} . \tag{3.13}$$

In particular, the matrix

$$\begin{pmatrix} A_{22} & P_2 \\ 0 & S \end{pmatrix} . \tag{3.14}$$

is a matrix representation of the linear mapping $(A_e + B_eF_e)|_{V_e^*}$.

We now show that certain properties of this matrix influence the solvability of the regulator equations.

Proposition 3.3 *Suppose* $\ker(B) = 0$ *and* $V^* \cap \operatorname{im}(B) = \{0\}$. *The regulator equations (3.4) are solvable if and only if the matrices*

$$\begin{pmatrix} A_{22} & P_2 \\ 0 & S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_{22} & 0 \\ 0 & S \end{pmatrix}$$

are similar.

Remark. Note that, in geometric terms, the condition indicated in the this statement can be expressed in the following way: the linear mapping $(A_e + B_eF_e)|_{V_e^*}$ has two complementary invariant subspaces, the restrictions of $(A_e + B_eF_e)|_{V_e^*}$ to these two subspaces being isomorphic to $(A + BF)|_{V^*}$ and, respectively, to the linear mapping represented by the matrix S. ◁

Proof. Necessity. Let Π, Γ be solutions of (3.4). Consider again matrix (3.12) and observe that

$$\begin{pmatrix} I & -\Pi & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} A + BF - \lambda I & P & B \\ 0 & S - \lambda I & 0 \\ C & Q & 0 \end{pmatrix} \begin{pmatrix} I & 0 & \Pi \\ 0 & 0 & I \\ 0 & I & \Gamma - F\Pi \end{pmatrix}$$
$$= \begin{pmatrix} A + BF - \lambda I & B & 0 \\ C & 0 & 0 \\ 0 & 0 & S - \lambda I \end{pmatrix} .$$

Since these transformations leave the invariant polynomials unchanged, it is concluded that the invariant polynomials of (3.11), i.e. those of

$$\begin{pmatrix} A_{22} - \lambda I & P_2 \\ 0 & S - \lambda I \end{pmatrix},$$

coincide with those of

$$\begin{pmatrix} A + BF - \lambda I & B & 0 \\ C & 0 & 0 \\ 0 & 0 & S - \lambda I \end{pmatrix}.$$

The latter, as in the proof of the previous Proposition, can be shown to coincide with those of

$$\begin{pmatrix} A_{22} - \lambda I & 0 \\ 0 & S - \lambda I \end{pmatrix},$$

and this completes the proof of the necessity.

Sufficiency. Consider the partition of x_e as $x_e = \mathrm{col}(x_1, x_2, w)$. The previous discussion has shown that the subspace V_e^* has dimension

$$n_e^* = dim(x_2) + dim(w)$$

and also that any vector of the form $\mathrm{col}(0, x_2, 0)$ is in V_e^*. Moreover, it is possible to show that no vector of the form $\mathrm{col}(x_1, 0, 0)$ can be in V_e^*, because otherwise the vector $\mathrm{col}(x_1, 0)$ would be in V^* and this is a contradiction. Thus, V_e^* is spanned by the columns of a matrix of the form

$$\begin{pmatrix} 0 & \Pi_1 \\ I & 0 \\ 0 & I \end{pmatrix}.$$

Since, for some F_e, $(A_e + B_eF_e)V_e^* \subset V_e^*$, and (3.14) is a matrix representation of the linear mapping $(A_e + B_eF_e)|_{V_e^*}$, it is deduced that there exist matrices F_1, F_2, L such that

$$\begin{pmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 & P_1 + B_1L \\ A_{21} & A_{22} & P_2 \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} 0 & \Pi_1 \\ I & 0 \\ 0 & I \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \Pi_1 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{22} & P_2 \\ 0 & S \end{pmatrix}.$$

This identity implies

$$\begin{array}{rcl} A_{12} + B_1 F_2 & = & 0 \\ A_{11}\Pi_1 + B_1(F_1\Pi_1 + L) + P_1 & = & \Pi_1 S \\ A_{21}\Pi_1 & = & 0\,. \end{array} \tag{3.15}$$

Moreover, since $V_e^* \subset \ker(C_e)$, Π_1 also satisfies

$$0 = C_1\Pi_1 + Q\,. \tag{3.16}$$

If the two matrices in the statement of the Proposition are similar, there exist a matrix Π_2 satisfying

$$A_{22}\Pi_2 + P_2 = \Pi_2 S\,. \tag{3.17}$$

Identities (3.15), (3.16) and (3.17) altogether show that

$$\Pi = \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}, \qquad \Gamma = F_2\Pi_2 + F_1\Pi_1 + L$$

solve the regulator equations (3.4). ◁

3.2 The zero dynamics of a nonlinear system

We review in this section[1] the notion of zero dynamics of a nonlinear system. For simplicity, we restrict ourselves to the particular case in which the system in question is *affine* in the input, i.e. is modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3.18}$$

in which $x \in X \subset \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $f(0) = 0$ and $h(0) = 0$.

Let M be a smooth connected submanifold of x which contains the point $x = 0$. The manifold M is said to be *locally controlled invariant* if there exist a smooth mapping $u : M \to \mathbb{R}^m$, and a neighborhood U of the origin in $\mathbb{R}^n$, such that the vector field $\tilde{f}(x) = f(x) + g(x)u(x)$ is *tangent* to M for all $x \in M \cap U$.

An *output zeroing submanifold* is a connected submanifold M of X which contains the origin and satisfies the following:

[1] This and the following sections are taken from [34].

(i) M is locally controlled invariant;

(ii) $h(x) = 0$ for all $x \in M$.

In other words, an output zeroing submanifold is a submanifold M of the state space with the property that - for some choice of feedback control $u(x)$ - the trajectories of the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u(x) \\ y &= h(x) \end{aligned} \tag{3.19}$$

which start in M stay in M for some interval of time, and the corresponding output is identically zero in the meanwhile.

If M and M' are connected smooth submanifolds of X which both contain the point $x = 0$, we say that M locally contains M' (or coincides with M') if for some neighborhood U of the origin $M \cap U \supset M' \cap U$ (or $M \cap U = M' \cap U$). An output zeroing submanifold M is *locally maximal* if, for some neighborhood U of the origin, any other output zeroing submanifold M' satisfies $M \cap U \supset M' \cap U$. The construction of a locally maximal output zeroing submanifold is illustrated in the following statement.

Proposition 3.4 *Part 1: Define a nested sequence of subsets* $M_0 \supset M_1 \supset \ldots$ *of* X *in the following way. Set* $M_0 = \{x \in X : h(x) = 0\}$. *At each* $k > 0$, *suppose that, for some neighborhood* U_{k-1} *of* 0, $M_{k-1} \cap U_{k-1}$ *is a smooth manifold, let* $\tilde{M}_{k-1}$ *denote the connected component of* $M_{k-1} \cap U_{k-1}$ *which contains the origin* ($\tilde{M}_{k-1}$ *is nonempty because* $f(0) = 0$) *and define* M_k *as*

$$M_k = \{x \in \tilde{M}_{k-1} : \quad f(x) \in \text{span}\{g_1(x), \ldots, g_m(x)\} + T_x\tilde{M}_{k-1}\}.$$

Then, for some $k^* \geq 0$ *and some neighborhood* U_{k^*} *of* 0, $M_{k^*+1} = \tilde{M}_{k^*}$. *Suppose also that*

$$\begin{aligned} &\dim(\text{span}\{g_1(x), \ldots, g_m(x)\}) = \text{constant} \\ &\dim(\text{span}\{g_1(x), \ldots, g_m(x)\} \cap T_x\tilde{M}_{k^*}) = \text{constant} \end{aligned}$$

for all $x \in \tilde{M}_k$. *Then, the manifold* $Z^* = \tilde{M}_{k^*}$ *is a locally maximal output zeroing submanifold.*

Part 2: If, in addition,

$$\begin{aligned} &\dim(\text{span}\{g_1(x), \ldots, g_m(x)\}) = m \\ &\text{span}\{g_1(x), \ldots, g_m(x)\} \cap T_xZ^* = 0 \end{aligned} \tag{3.20}$$

at $x = 0$, then there exists a unique smooth mapping $u^* : Z^* \to \mathbb{R}^m$ such that the vector field

$$f^*(x) = f(x) + g(x)u^*(x)$$

is tangent to Z^* .

Suppose the hypotheses listed in the previous Proposition are satisfied. Since the vector field $f^*(x)$ is tangent to Z^*, the *restriction* of $f^*(x)$ to Z^* is a well-defined vector field of Z^*. In what follows, unless otherwise specified, by $f^*(x)$ we will always indicate–with some abuse of notation–the restriction of $f^*(x)$ to Z^*. The submanifold Z^* is called the (local) *zero dynamics submanifold* and the vector field $f^*(x)$ is called the *zero dynamics vector field.* The pair (Z^*, f^*) is called the *zero dynamics* of the system (3.18).

By construction, the dynamical system

$$\dot{x} = f^*(x), \qquad x \in Z^*$$

identifies the internal dynamical behavior induced on the system when the output has been forced, by proper choice of initial state and input, to remain zero for some interval of time.

Remark. In the case of a linear system, all the hypotheses of Proposition 3.4, Part 1, which lead to the existence of the manifold Z^*, are always satisfied. The latter coincides with V^*, the largest controlled invariant subspace of $ker(C)$, namely the largest subspace of $ker(C)$ satisfying

$$AV^* \subset V^* + \mathrm{im}(B).$$

The hypotheses (3.20) of Proposition 3.4, Part 2, reduce to the following ones:

$$\dim(\mathrm{im}(B)) = m, \qquad V^* \cap \mathrm{im}(B) = 0 \tag{3.21}$$

which, as mentioned in section 3.1, are exactly the conditions under which the transfer function matrix of the system is invertible. The state feedback $u^*(x)$ which renders $f^*(x)$ tangent to V^* is a linear function of x, namely $u^*(x) = Fx$, and by construction the subspace V^* is *invariant* under the linear mapping $(A + BF)$. In case assumptions (3.21) hold, the restriction $F|_{V^*}$ is unique. ◁

3.3 Existence of solutions of the nonlinear regulator equations

To the purpose of determining the solution of the nonlinear regulator equations (3.1), the concepts summarized in the previous section can be used in the following way. Suppose the controlled plant in (3.2) is affine in both the control and in the disturbance input, and observe that system (3.2), which will be henceforth denoted as system Σ_e, can be put in the form

$$\begin{aligned}\dot{x}_e &= f_e(x_e) + g_e(x_e)u \\ e &= h_e(x_e),\end{aligned} \tag{3.22}$$

where

$$x_e = \begin{pmatrix} x \\ w \end{pmatrix}, \quad f_e(x_e) = \begin{pmatrix} f(x) + p(x)w \\ s(w) \end{pmatrix}, \quad g_e(x_e) = \begin{pmatrix} g(x) \\ 0 \end{pmatrix},$$
$$h_e(x_e) = h(x,w) \quad . \tag{3.23}$$

In the light of the above, equations (3.1) can be rewritten as

$$\begin{aligned}\frac{\partial \pi}{\partial w}s(w) &= f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \\ 0 &= h(\pi(w),w) \ .\end{aligned} \tag{3.24}$$

Suppose that conditions (3.24) are satisfied, and consider, in the state-space $X_e = X \times W$ of Σ_e, the submanifold

$$M_s = \{(x,w) \in X_e : \quad x = \pi(w)\}. \tag{3.25}$$

In view of the terminology introduced in the previous section, we may easily observe that the manifold thus defined is an *output zeroing submanifold* of the system Σ_e. In fact, the first one of (3.24) exactly says that M_s is locally controlled invariant (invariance is achieved under the feedback law $u = c(w)$), and the second one of (3.24) says that M_s is annihilated by the "output" map $e = h_e(x_e)$.

The system in question may have also another relevant output zeroing submanifold. Let Σ denote the system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x,0),\end{aligned} \tag{3.26}$$

(with $f(x)$, $g(x)$, and $h(x,0)$ the same as in (3.23), that is in Σ_e) and suppose Σ satisfies the assumption of Proposition 3.4, so that a zero dynamics (Z^*, f^*) can be defined in a neighborhood $U \subset X$ of 0. Let $u^*(x)$ denote the (unique) feedback law which renders $f^*(x) = f(x) + g(x)u^*(x)$ tangent to Z^*, and consider, in the state-space $X_e = X \times W$ of Σ_e, the submanifold

$$M_z = Z^* \times \{0\}. \tag{3.27}$$

This submanifold is indeed an output zeroing submanifold of Σ_e. For, this manifold is locally controlled invariant (invariance is achieved under the feedback law $u = u^*(x)$ because $w = 0$ is an equilibrium of $\dot{w} = s(w)$) and is also annihilated by the output map (because $h(x,0) = 0$ for each $x \in Z^*$).

In the next statement, we illustrate more precisely the relation between the zero dynamics of Σ_e and those of Σ. Of course, our analysis requires the existence of both the zero dynamics in question, so we assume

Z1: the system Σ satisfies the hypotheses of Proposition 3.4, so that a zero dynamics (Z^*, f^*) can be defined in a neighborhood $U \subset X$ of 0;

Z2: the system Σ_e satisfies the hypotheses of the (corresponding) Proposition 3.4, so that a zero dynamics (Z_e^*, f_e^*) can be defined in a neighborhood $U_e \subset X_e$ of 0.

Lemma 3.5 *Suppose hypotheses Z1 and Z2 hold. Suppose that, in a neighborhood of $x_e = 0$, the set*

$$M = Z_e^* \cap (X \times \{0\}) \tag{3.28}$$

is a smooth submanifold. Then M locally coincides with the submanifold M_z defined by (3.27). Moreover, M is locally invariant under f_e^*, and the restriction of f_e^* to M is locally diffeomorphic to the vector field f^* which characterizes the zero dynamics of Σ.

Proof. Since Z_e^* is a locally maximal output zeroing submanifold for Σ_e, Z_e^* locally contains the submanifold M_z, which is an output zeroing submanifold for Σ_e. Since by construction $M_z \subset (X \times \{0\})$, we deduce that M locally contains M_z. Observe now that, at each $(x,0) \in M$, $h(x,0) = 0$. In fact, $h(x,w) = 0$ at each $(x,w) \in$

Z_e^*. Moreover, if we let $u_e^*(x,w)$ denote the (unique) input which constrains the flow of Σ_e to evolve on Z_e^*, we immediately see that the input $u = u_e^*(x,0)$ constrains the flow of (3.26) to evolve on M (because $w = 0$ is an equilibrium of the exosystem). From this we deduce that M is an output zeroing submanifold for Σ. M must be locally contained in M_z, and therefore M locally coincides with M_z.

The second part of the statement is proved in this way. At each point $(x,0)$ of M_z, the input $u^*(x)$ renders the vector $f(x)+g(x)u^*(x)$ tangent to M_z. Σ_e satisfies the assumptions of Proposition 3.4 (in particular (3.20)) and, at each $x_e \in Z_e^*$ there is a unique u_e^* such that $f_e^* = f_e + g_e u_e^*$ is tangent to Z_e^*. Since $M_z \subset Z_e^*$, we deduce that $u^*(x) = u_e^*(x,0)$ for each $x \in M_z$. The immersion

$$\begin{aligned} \sigma: \quad & Z^* \to X_e \\ & x \to (x,0) \end{aligned}$$

is a diffeomorphism of Z^* onto M_z, and

$$\sigma_* f^*(x) = (f_e + g_e u_e^*) \circ \sigma(x) = (f_e^*) \circ \sigma(x).$$

Thus, the restriction of f_e^* to M_z is locally diffeomorphic to the vector field f^*. $\triangleleft$

The next lemma illustrates, in terms of properties of the zero dynamics of Σ_e, necessary conditions for the solvability of the regulator equations (3.24).

Lemma 3.6 *Suppose hypotheses Z1 and Z2 hold. Suppose there exist smooth mappings $x = \pi(w)$, with $\pi(0) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying conditions (3.24). Then:*

i) in a neighborhood of $x_e = 0$, the set M defined by (3.28) is a smooth submanifold,

ii) Z_e^* locally contains the submanifold M_s, defined by (3.25), and

$$T_0 Z_e^* = T_0 M_s \oplus T_0 M \ ,$$

iii) M_s is locally invariant under f_e^*, and the restriction of f_e^* to Z_s is locally diffeomorphic to the vector field $s(w)$ which characterizes the exosystem.

Proof. Since Z_e^* is a locally maximal output zeroing submanifold for Σ_e, Z_e^* locally contains M_z and M_s, which are output zeroing submanifolds for Σ_e. At $(x,w) = (0,0)$, $T_0M_s \oplus T_0(X \times \{0\}) = T_0X_e$. Thus, since $T_0M_s \subset T_0Z_e^*$, we have $T_0Z_e^* + T_0(X \times \{0\}) = T_0X_e$. By continuity, $T_{(x,w)}Z_e^*$ and $T_{(x,w)}(X \times \{0\})$ span $T_{(x,w)}X_e$ for all $(x,w) \in Z_e^* \cap (X \times \{0\})$ in a neighborhood of $(0,0)$ and this implies i).

By definition of M,

$$\dim(Z_e^*) \leq s + \dim(M)$$

and therefore, since $s = \dim(M_s)$, we obtain

$$\dim(Z_e^*) \leq \dim(M_s) + \dim(M).$$

Also the reverse inequality holds because $T_0M_s \cap T_0M = \{0\}$, and this proves ii).

By the first one of (3.24), at each point $(\pi(w), w)$ of M_s, the input $u = c(w)$ renders the vector

$$f_e(\pi(w), w) + g_e(\pi(w), w)c(w)$$

tangent to M_s. Thus, again appealing to the uniqueness of u_e^*, we deduce that $c(w)$ necessarily coincides with $u_e^*(\pi(w), w)$. The immersion

$$\begin{aligned} \mu : \quad & W^\circ \to X_e \\ & w \to (\pi(w), w) \end{aligned}$$

induces a local diffeomorphism of a neighborhood of the origin in W° onto a neighborhood of the origin in M_s. Again by (3.24) (with $c(w)$ replaced by $u_e^*(\pi(w), w)$) we have

$$\mu_* s(w) = (f_e + g_e u_e^*) \circ \mu(w) = (f_e^*) \circ \mu(w).$$

Thus, the restriction of f_e^* to M_s is locally diffeomorphic to the vector field $s(w)$, i.e., iii) holds. ◁

We are now in a position to prove a nonlinear analog of Proposition 3.3.

Theorem 3.7 *Suppose hypotheses Z1, Z2 hold. Let (Z_e^*, f_e^*) denote the zero dynamics of Σ_e. Then there exist smooth mappings $x = \pi(w)$, with $\pi(0) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying equations (3.24), if and only if the zero dynamics of Σ_e have the following properties:*

i) in a neighborhood of $x_e = 0$, the set M defined by (3.28) is a smooth submanifold,

ii) there exists a submanifold Z_s of Z_e^*, of dimension s, which contains the origin, such that

$$T_0 Z_e^* = T_0 Z_s \oplus T_0 M ,$$

iii) Z_s is locally invariant under f_e^*, and the restriction of f_e^* to Z_s is locally diffeomorphic to the vector field $s(w)$ which characterizes the exosystem.

Proof. The necessity immediately follows from Lemma 3.6. In order to prove the sufficiency, we first show that Z_s can be locally expressed in the form of the graph of a smooth mapping $x = \gamma(w)$ defined in a neighborhood $W^\circ \subset W$ of 0. Note that, by definition, $T_0 Z_e^* \cap T_0(X \times \{0\}) = T_0 M$, and therefore ii) implies

$$T_0 Z_s \cap T_0(X \times \{0\}) = \{0\}. \tag{3.29}$$

By assumption, Z_s is an s-dimensional submanifold; thus, there is a smooth mapping

$$\phi : \quad \tilde{X}_e \to \mathbb{R}^n$$

(where $\tilde{X}_e$ is a neighborhood of 0 in X_e), such that

$$Z_s = \{(x, w) \in \tilde{X}_e : \quad \phi(x, w) = 0\}.$$

Condition (3.29) implies that the matrix $(\partial\phi/\partial x)$ is invertible at 0 and, by the implicit function theorem, there exists a smooth mapping $x = \gamma(w)$ whose graph coincides with Z_s in a neighborhood of 0.

We now prove that iii) implies the fulfillment of conditions (3.24). As in the proof of Lemma 3.6, consider the immersion

$$\begin{aligned} \mu : \quad & W^\circ \to X_e \\ & w \to (\gamma(w), w) \end{aligned} \tag{3.30}$$

which induces a local diffeomorphism of the neighborhood of the origin in W° onto a neighborhood of the origin in Z_s. By iii), f_e^* is tangent to Z_s, and there is a diffeomorphism $\psi : \quad W^\circ \to W^\circ$, such that

$$(\mu \circ \psi)_* s(w) = (f_e^*) \circ \mu \circ \psi(w) = (f_e + g_e u_e^*) \circ \mu \circ \psi(w).$$

This clearly shows that the mappings

$$\begin{aligned} \pi(w) &= \gamma(w) \\ c(w) &= u_e^*(\gamma(w), w) \end{aligned} \tag{3.31}$$

satisfy the first one of (3.24). The fulfillment of the second one of (3.24) is a straightforward consequence of the fact that Z_s is a submanifold of Z_e^* and the latter is annihilated by $h_e(x_e)$. ◃

Remark. Note that condition i), by Lemma 3.5, implies that M is locally invariant under f_e^*, and the restriction of f_e^* to M is locally diffeomorphic to the vector field $f^*(x)$ which characterizes the zero dynamics of Σ. Thus, (3.24) are solvable if and only if the zero dynamics of Σ_e possess two complementary invariant submanifolds, the flows of f_e^* on these two submanifolds being diffeomorphic to that of the exosystem and, respectively, to that of the zero dynamics of Σ. ◃

The previous Theorem clearly shows that the solvability of the regulator equations is a property of the zero dynamics of Σ_e. In particular, we stress the fact that the proof of this Theorem demonstrates the existence of a smooth mapping $x = \pi(w)$, which solves (3.24) together with the mapping

$$c(w) = u_e^*(\pi(w), w)$$

where u_e^* is the unique input which renders f_e^* tangent to Z_e^*. In other words, if the zero dynamics of Σ_e are known (and so is the input u_e^*), the only real problem to deal with–in order to be able to solve a regulator equations–is the one of finding a submanifold Z_s of Z_e^* with the following properties:

- Z_s has dimension s;
- Z_s is transverse to $X \times \{0\}$;
- Z_s is locally invariant under the zero dynamics vector field f_e^*;
- the restriction of f_e^* to Z_s is diffeomorphic to the vector field $s(w)$ which characterizes the exosystem.

We hereinafter present a useful sufficient condition for the existence of such a submanifold.

Corollary 3.8 *Suppose hypotheses Z1, Z2 hold. Let (Z_e^*, f_e^*) denote the zero dynamics of Σ_e. Then there exist smooth mappings $x = \pi(w)$, with $\pi(0) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying equations (3.24), if*

i) $T_0 Z_e^* + T_0(X \times \{0\}) = T_0 X_e$,
ii) the zero dynamics of Σ *have a hyperbolic equilibrium at* $x = 0$.

Proof. As already observed in the proof of Lemma 3.6, i) implies that in a neighborhood of the origin, M is a smooth submanifold. Thus, by Lemma 3.5, M locally coincides with $M_z = Z^* \times \{0\}$. As a consequence, the vector field

$$f_e^*(x,w) = \mathrm{col}(f(x) + p(x)w + g(x)u_e^*(x,w), s(w))$$

at each point $(x,0)$ of M coincides with the vector field $col(f^*(x),0)$, where $f^*(x)$ is the zero dynamics vector field of Σ. Set

$$A = \left[\frac{\partial f^*(x)}{\partial x}\right]_{(0)}, \qquad A_e = \left[\frac{\partial f_e^*(x,w)}{\partial x} \, \frac{\partial f_e^*(x,w)}{\partial w}\right]_{(0,0)}$$

and note that

$$A_e = \begin{pmatrix} A & * \\ 0 & S \end{pmatrix}.$$

The restriction A^* of A to its invariant subspace $T_0 Z^*$ characterizes the linear approximation of the zero dynamics of Σ, and the restriction A_e^* of A_e to its invariant subspace $T_0 Z_e^*$ characterizes the linear approximation of the zero dynamics of Σ_e. Note that $T_0(Z^* \times \{0\})$ is a subspace of $T_0 Z_e^*$, with codimension s. Thus, the spectrum of A_e^* is the disjoint union of the spectra of A^* and of S. By assumption ii), A^* has no eigenvalue on the imaginary axis, while the eigenvalues of S are on the imaginary axis. From the center manifold theorem, we deduce that there is a C^k $(k \geq 2)$ center manifold Z_s for Z_e^*, which indeed satisfies conditions ii) and iii) of Theorem 3.7. This manifold can be expressed as a graph of a mapping $x = \pi(w)$, which satisfies (3.24) for $c(w) = u_e^*(\pi(w), w)$. ◁

Chapter 4

Robust Output Regulation

4.1 Structurally stable local regulation

The purpose of this Chapter is to study problems of output regulation in the presence of parameter uncertainties. In order to facilitate the exposition of the material, we proceed by addressing problems of increasing complexity, beginning with the solution of a problem of local output regulation in the presence of small parameter variations, then continuing with the solution of a problem of local output regulation in the presence of parameter variations ranging on prescribed sets, and then ending with design of a controller solving the problem of output regulation for any initial condition over an arbitrarily large (but fixed) compact set, robustly with respect to unknown parameters also ranging over an arbitrarily large (but fixed) compact set.

For convenience, we continue to consider nonlinear plants modeled by equations of the form (1.1), in which we explicitly introduce a vector $\mu \in \mathbb{R}^p$ of unknown parameters, that is

$$\begin{aligned} \dot{x} &= f(x, u, w, \mu) \\ e &= h(x, w, \mu). \end{aligned} \tag{4.1}$$

Without loss of generality, we suppose $\mu = 0$ is the nominal value of the uncertain and, for consistency with the analysis developed earlier in this book, we assume $f(x, u, w, \mu)$ and $h(x, w, \mu)$ to be C^k functions of their arguments. Moreover, we also assume $f(0,0,0,\mu) = 0$ and $h(0,0,\mu) = 0$ for each value of μ.

The problem addressed in this section can be described as follows.

Structurally stable local output regulation. Given a nonlinear system of the form (4.1) with exosystem

$$\dot{w} = s(w)\,, \tag{4.2}$$

find a controller of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, e) \\ u &= \theta(\xi) \end{aligned} \tag{4.3}$$

such that, for some neighborhood $\mathcal{P}$ of $\mu = 0$ in $\mathbb{R}^p$ and for each $\mu \in \mathcal{P}$:

(a) the equilibrium $(x, \xi) = (0,0)$ of the unforced closed loop system

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi), 0, \mu) \\ \dot{\xi} &= \eta(\xi, h(x, 0, \mu)) \end{aligned}$$

is locally asymptotically stable in the first approximation,

(b) the forced closed loop system

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi), w, \mu) \\ \dot{\xi} &= \eta(\xi, h(x, w, \mu)) \\ \dot{w} &= s(w) \end{aligned}$$

is such that

$$\lim_{t\to\infty} e(t) = 0$$

for each initial condition $(x(0), \xi(0), w(0))$ in a neighborhood of the equilibrium $(0,0,0)$.

The solution of the problem of structurally stable output regulation happens to be a straightforward consequence of the theory of local output regulation developed in section 2.3. As a matter of fact, in order to address such a problem it suffices to look at w and μ as if they were components of an "augmented" exogenous input w^{a}, which in this case (since the parameter μ ius assumed to be constant) would be generated by the "augmented" exosystem

$$\dot{w}^{a} = s^{a}(w^{a}) = \begin{pmatrix} s(w) \\ 0 \end{pmatrix}.$$

With this notation, the "family" of plants (4.1) can be viewed as a single plant modeled by equations of the form (1.1), namely

$$\begin{aligned}\dot{x} &= f^{\mathrm{a}}(x, u, w^{\mathrm{a}})\\ e &= h^{\mathrm{a}}(x, w^{\mathrm{a}}).\end{aligned}$$

It is easy to realize that a controller which solves a standard problem of local output regulation for the "augmented" plant thus defined also solves the problem of structurally stable local regulation for the family of (4.1). In fact, suppose a controller of the form (4.3) is such that:

(a') the equilibrium $(x, \xi) = (0, 0)$ of the unforced closed loop system

$$\begin{aligned}\dot{x} &= f^{\mathrm{a}}(x, \theta(\xi), 0)\\ \dot{\xi} &= \eta(\xi, h^{\mathrm{a}}(x, 0)\end{aligned}$$

is locally asymptotically stable in the first approximation,

(b') the forced closed loop system

$$\begin{aligned}\dot{x} &= f^{\mathrm{a}}(x, \theta(\xi), w^{\mathrm{a}})\\ \dot{\xi} &= \eta(\xi, h^{\mathrm{a}}(x, w^{\mathrm{a}}))\\ \dot{w} &= s(w)\end{aligned}$$

is such that

$$\lim_{t\to\infty} e(t) = 0$$

for each initial condition $(x(0), \xi(0), w^{\mathrm{a}}(0))$ in a neighborhood of $(0, 0, 0)$.

Then, for some neighborhood $\mathcal{P}$ of $\mu = 0$ in $\mathbb{R}^p$, both requirements (a) and (b) of the previous definition hold (the former because stability in the first approximation is conserved under small parameter variations and the latter because in (b') the asymptotic decay of the error is assumed to hold - in particular - for all $\mu(0)$ in some neighborhood of $mu = 0$).

In this setup, it is easy to adapt the general necessary and sufficient conditions presented in Chapter 2, to yield a set of necessary and sufficient conditions for the solution of the problem of structurally stable local output regulation. Set

$$A(\mu) = \left[\frac{\partial f}{\partial x}\right]_{(0,0,0,\mu)}, \quad B(\mu) = \left[\frac{\partial f}{\partial u}\right]_{(0,0,0,\mu)}, \quad C(\mu) = \left[\frac{\partial h}{\partial x}\right]_{(0,0,\mu)}.$$

and observe that, because of the special form of the vector field $s^{\mathrm{a}}(w^{\mathrm{a}})$,

$$\frac{\partial \pi^{\mathrm{a}}}{\partial w^{\mathrm{a}}} s^{\mathrm{a}}(w^{\mathrm{a}}) = \frac{\partial \pi^{\mathrm{a}}(w,\mu)}{\partial w} s(w) \, .$$

Then, the following result holds.

Theorem 4.1 *Consider the plant (4.1) with exosystem (4.2). Suppose the exosystem is neutrally stable. The problem of structurally stable local output regulation is solvable if and only if there exist mappings* $x = \pi^{\mathrm{a}}(w,\mu)$ *and* $u = c^{\mathrm{a}}(w,\mu)$, *with* $\pi^{\mathrm{a}}(0,\mu) = 0$ *and* $c^{\mathrm{a}}(0,\mu) = 0$, *both defined in a neighborhood* $W^\circ \times \mathcal{P} \subset W \times \mathbb{R}^p$ *of the origin, satisfying the conditions*

$$\begin{aligned} \frac{\partial \pi^{\mathrm{a}}(w,\mu)}{\partial w} s(w) &= f(\pi^{\mathrm{a}}(w,\mu), c^{\mathrm{a}}(w,\mu), w, \mu) \\ 0 &= h(\pi^{\mathrm{a}}(w,\mu), w, \mu) \end{aligned} \tag{4.4}$$

for all $(w,\mu) \in W^\circ \times \mathcal{P}$ *and such that the autonomous system with output*

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\mu} &= 0 \\ u &= c^{\mathrm{a}}(w,\mu) \, , \end{aligned}$$

is immersed into a system

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ u &= \gamma(\xi) \, , \end{aligned}$$

defined on a neighborhood Ξ° *of the origin in* $\mathbb{R}^\nu$, *in which* $\varphi(0) = 0$ *and* $\gamma(0) = 0$ *and the two matrices*

$$\Phi = \left[\frac{\partial \varphi}{\partial \xi}\right]_{\xi=0}, \qquad \Gamma = \left[\frac{\partial \gamma}{\partial \xi}\right]_{\xi=0}$$

are such that the pair

$$\begin{pmatrix} A(0) & 0 \\ NC(0) & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B(0) \\ 0 \end{pmatrix}$$

is stabilizable for some choice of the matrix N, *and the pair*

$$\begin{pmatrix} C(0) & 0 \end{pmatrix}, \qquad \begin{pmatrix} A(0) & B(0)\Gamma \\ 0 & \Phi \end{pmatrix}$$

is detectable.

Remark. Note that the first condition indicated in this Theorem, namely the existence of a solution $\pi^{a}(w,\mu), c^{a}(w,\mu)$ of the equations (4.4) for each μ in a neighborhood of $\mu = 0$ is a trivial necessary condition for the existence of a solution of the problem of structurally stable local output regulation. The condition in question, in fact, is one of the necessary conditions (see Theorem 2.5) for the existence of a solution of the standard problem of output regulation for any *fixed* value of μ. ◁

Remark. The condition that system $\{W^{\circ} \times \mathcal{P}, s^{a}, c^{a}\}$ is immersed into a system $\{\Xi^{\circ}, \varphi, \gamma\}$ is the existence of a mapping $\tau^{a}(w,\mu)$ such that

$$\frac{\partial \tau^{a}}{\partial w} s(w) = \varphi(\tau^{a}(w,\mu)), \qquad c^{a}(w,\mu) = \gamma(\tau^{a}(w,\mu)).$$

Choose K, L, M, N as suggested in the proof of Theorem 2.5. Then, a simple calculation shows that

$$M_{c} = \{(x,\xi_0,\xi_1,w,\mu) : x = \pi^{a}(w,\mu), \xi_0 = 0, \xi_1 = \tau^{a}(w,\mu)\}$$

is a center manifold for the system

$$\begin{aligned}
\dot{x} &= f(x, M\xi_0 + \gamma(\xi_1), w, \mu)\\
\dot{\xi}_0 &= K\xi_0 + Lh(x,w,\mu)\\
\dot{\xi}_1 &= \varphi(\xi_1) + Nh(x,w,\mu)\\
\dot{w} &= s(w)\\
\dot{\mu} &= 0.
\end{aligned}$$

at the equilibrium $(x,\xi_0,\xi_1,w,\mu) = (0,0,0,0,0)$. Since a center manifold contains all other equilibria which are sufficiently close to this particular one, it is deduced that any point $(x,\xi_0,\xi_1,w,\mu) = (0,0,0,0,\mu)$ is a point of M_c. In particular, $\tau^{a}(0,\mu) = 0$. ◁

The relevance of the notion of *immersion* in the solution of a problem of output regulation is perhaps best motivated by the following arguments. Observe that the linear approximation of

$$\begin{aligned}
\dot{w} &= s(w)\\
\dot{\mu} &= 0u = c^{a}(w,\mu)\,,
\end{aligned}$$

at the equilibrium $(w,\mu) = (0,0)$ *cannot* be detectable. In fact, since $c^{a}(0,\mu) = 0$ by hypothesis,

$$\frac{\partial}{\partial \mu} c^{a}(0,\mu) = 0\,,$$

and the linear approximation in question is characterized by a pair of matrices of the form

$$\begin{pmatrix} * & 0 \end{pmatrix}, \qquad \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$

which is indeed not detectable. Thus, it is not possible to have the conditions of the Theorem directly satisfied by the trivial immersion of this system into itself. However, the system in question may well be immersed into *another* system of the form

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ u &= \gamma(\xi) , \end{aligned}$$

having a detectable approximation at $\xi = 0$. This happens, for instance, in all cases in which the exosystem and the function $c^{\mathrm{a}}(w, \mu)$ are such that conditions for corresponding to those described in the Corollaries of Theorem 2.5 hold.

We limit hereafter to describe one of these cases. Observe that, because of the special form of the vector field $s^{\mathrm{a}}(w^{\mathrm{a}})$, the derivative of any function $\lambda(w, \mu)$ along $s^{\mathrm{a}}(w^{\mathrm{a}})$ reduces to

$$L_{s^{\mathrm{a}}}\lambda(w, \mu) = \frac{\partial \lambda(w, \mu)}{\partial w} s(w) .$$

For convenience, the latter will be simply indicated as

$$L_s \lambda(w, \mu) .$$

Corollary 4.2 *Consider the plant (4.1), with exosystem (4.2). Suppose the exosystem is neutrally stable. Suppose the pair* $(A(0), B(0))$ *is stabilizable and the pair* $(C(0), A(0))$ *is detectable. Suppose there exist mappings* $x = \pi^{\mathrm{a}}(w, \mu)$ *and* $u = c^{\mathrm{a}}(w, \mu)$*, with* $\pi^{\mathrm{a}}(0, \mu) = 0$ *and* $c^{\mathrm{a}}(0, \mu) = 0$*, both defined in a neighborhood* $W^{\circ} \times \mathcal{P} \subset W \times \mathbb{R}^p$ *of the origin, satisfying the conditions (4.4). Suppose also there exist integers* $p_1, \ldots, p_m$ *and functions*

$$\begin{array}{rcl} \phi_i \quad : \mathbb{R}^{p_i} & \to & \mathbb{R} \\ (\zeta_1, \ldots, \zeta_i) & \mapsto & \phi_i(\zeta_1, \ldots, \zeta_i) \end{array}$$

such that, for all $1 \le i \le m$*, the i-th component* $c_i^{\mathrm{a}}(w, \mu)$ *of* $c^{\mathrm{a}}(w, \mu)$ *satisfies*

$$L_s^{p_i} c_i^{\mathrm{a}}(w, \mu) = \phi_i(c_i^{\mathrm{a}}(w, \mu), L_s c_i^{\mathrm{a}}(w, \mu), \ldots, L_s^{p_i - 1} c_i^{\mathrm{a}}(w, \mu)) , \quad (4.5)$$

for all $(w, \mu) \in W^\circ \times \mathcal{P}$. *Set*

$$d_{ij} = \left[\frac{\partial \phi_i}{\partial \zeta_j}\right]_{(0,\ldots,0)}$$

and

$$d_i(\lambda) = d_{i0} + d_{i1}\lambda + \ldots + d_{i,p_i-1}\lambda^{p_i-1} - \lambda^{p_i} .$$

Finally, suppose that the matrix

$$\begin{pmatrix} A(0) - \lambda I & B(0) \\ C(0) & 0 \end{pmatrix}$$

is nonsingular for every λ *which is a root of any of the polynomials* $d_1(\lambda), \ldots, d_m(\lambda)$ *having non-negative real part. Then there exists a controller which solves the problem of structurally stable local output regulation.*

4.2 Robust local regulation

In the previous section, we have established necessary and sufficient conditions for the existence of a controller of the form (4.3) which solves the problem of output regulation for any nonlinear system in the parametrized family (4.1), when the parameter μ ranges over *some* neighborhood $\mathcal{P}$ of $\mu = 0$ in the parameter space $\mathbb{R}^p$. We discuss now the problem of designing a controller which solves the problem in question when μ ranges over an *a priori fixed* compact set $\mathcal{P}^*$ in the parameter space.

To this end observe that if some fixed controller solves, for any μ in a preassigned set $\mathcal{P}^*$, the problem of local output regulation, then the necessary conditions of Theorem 4.1 must hold for every $\mu \in \mathcal{P}^*$. In particular, then, for every $\mu \in \mathcal{P}^*$ the equations (4.4) must have a solution $x = \pi^a(w, \mu)$ and $u = c^a(w, \mu)$, defined in a neighborhood W°_μ of the origin in W, and the autonomous system with output

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\mu} &= 0 \\ u &= c^a(w, \mu) , \end{aligned}$$

is immersed into a system

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ u &= \gamma(\xi) , \end{aligned}$$

defined on a neighborhood Ξ° of the origin in $\mathbb{R}^\nu$, with $\varphi(0) = 0$ and $\gamma(0) = 0$, and the pair of matrices

$$\Gamma = \left[\frac{\partial \varphi}{\partial \xi}\right]_{\xi=0}, \qquad \Phi = \left[\frac{\partial \gamma}{\partial \xi}\right]_{\xi=0}$$

is detectable.

Moreover, since the controller which solves the problem is required to stabilize for every value of μ the linear approximation of the plant at the equilibrium $(x, w) = (0, 0)$

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u \\ y &= C(\mu)x \,, \end{aligned} \tag{4.6}$$

the latter must be *robustly stabilizable* on $\mathcal{P}^*$, i.e. there must exist F, G, H such that

$$\begin{pmatrix} A(\mu) & B(\mu)H \\ GC(\mu) & F \end{pmatrix} \tag{4.7}$$

has all eigenvalues with negative real part for every $\mu \in \mathcal{P}^*$.

Suppose now all these conditions hold and consider again the controller described in section 2.3, in which we set

$$K = F, \qquad L = H, \qquad M = G$$

where F, G, H are matrices such that (4.7) has all eigenvalues with negative real part for every $\mu \in \mathcal{P}^*$. It is clear from all arguments introduced so far that if the controller thus defined

$$\begin{aligned} \dot{\xi}_0 &= F\xi_0 + He \\ \dot{\xi}_1 &= \varphi(\xi_1) + Ne \\ u &= G\xi_0 + \gamma(\xi_1) \,. \end{aligned}$$

is such that the associated closed loop system is stable in the first approximation (at the equilibrium $(x, \xi_0, \xi_1, w) = (0, 0, 0, 0)$) for every $\mu \in \mathcal{P}^*$, then this controller solves the problem of robust local output regulation.

Stability in the first approximation (at the equilibrium (x, ξ_0, ξ_1, w) $= (0, 0, 0, 0)$) of the closed loop system thus defined depends on the eigenvalues of the matrix

$$\begin{pmatrix} A(\mu) & B(\mu)H & B(\mu)\Gamma \\ GC(\mu) & F & 0 \\ NC(\mu) & 0 & \Phi \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} A(\mu) & B(\mu)H \\ GC(\mu) & F \end{pmatrix} & \begin{pmatrix} B(\mu) \\ 0 \end{pmatrix}\Gamma \\ N\begin{pmatrix} C(\mu) & 0 \end{pmatrix} & \Phi \end{pmatrix} \tag{4.8}$$

in which, the matrices F, G, H and Γ, Φ are fixed and, by hypothesis, the submatrix

$$\begin{pmatrix} A(\mu) & B(\mu)H \\ GC(\mu) & F \end{pmatrix}$$

has all eigenvalues with negative real part for every $\mu \in \mathcal{P}^*$.

The only degree of freedom left in the design of the controller is the matrix N, which must be chosen in such a way that all the eigenvalues of (4.8) negative real part for every $\mu \in \mathcal{P}^*$. The following results describes a simple sufficient condition under which this is possible, in the case of a single-input single-output system.

Lemma 4.3 *Consider the parametrized family of single-input single-output linear systems of the form (4.6) where μ ranges over a fixed compact subset $\mathcal{P}^*$ of $\mathbb{R}^p$, with $0 \in \mathcal{P}^*$. Suppose the feedback law*

$$\begin{aligned} \dot{\xi}_0 &= F\xi_0 + Gy \\ u &= H\xi_0 + v\,, \end{aligned} \tag{4.9}$$

is a robust stabilizer for (4.6). Let $T_\mu(s)$ denote the transfer function of the closed loop system (4.6)-(4.9), i.e. of the system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi}_0 \end{pmatrix} &= \begin{pmatrix} A(\mu) & B(\mu)H \\ GC(\mu) & F \end{pmatrix} \begin{pmatrix} x \\ \xi_0 \end{pmatrix} + \begin{pmatrix} B(\mu) \\ 0 \end{pmatrix} v \\ y &= \begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi_0 \end{pmatrix}. \end{aligned} \tag{4.10}$$

Suppose

$$\max_{\mu \in \mathcal{P}^*} \sup_{\omega \in \mathbb{R}} \left|\operatorname{Arg}\left(\frac{T_\mu(j\omega)}{T_0(j\omega)}\right)\right| < \frac{\pi}{2}\,. \tag{4.11}$$

Let

$$d(\lambda) = \lambda^q + a_{q-1}\lambda^{q-1} + \ldots + a_1\lambda$$

be a polynomial whose roots are purely imaginary and simple and suppose

$$\det \begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix} \neq 0$$

for every root λ of $d(\lambda)$ and all $\mu \in \mathcal{P}^*$. Set

$$\begin{aligned} \Phi &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_1 & -a_2 & \cdots & -a_{q-1} \end{pmatrix}, \\ \Gamma &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

Then, there exists a vector

$$N = \mathrm{col}(b_0, b_1, \ldots, b_{q-1})$$

such that the interconnection of (4.10) with

$$\begin{aligned} \dot{\xi}_1 &= \Phi\xi_1 + Ny \\ v &= \Gamma\xi_1 , \end{aligned} \tag{4.12}$$

is stable for every $\mu \in \mathcal{P}^*$.

Proof. Let $N_\mu(s)$ and $D_\mu(s)$ denote the numerator and, respectively, denominator of $T_\mu(s)$ and let $\bar{n}$ denote the degree of $D_\mu(s)$. By hypothesis, all roots of $D_\mu(s)$ are in $\mathbb{C}^-$, for every $\mu \in \mathcal{P}^*$. Also, since

$$N_\mu(s) = \bar{k}_\mu \det \begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix}$$

for some real number $\bar{k}_\mu$, none of the roots of $N_\mu(s)$ coincides with any root of $d(s)$, for every $\mu \in \mathcal{P}^*$.

Let

$$K(s) = k\frac{n(s)}{d(s)}$$

denote the transfer function of system (4.12), in which

$$d(s) = s\prod_{i=1}^{q-1}(s - \lambda_i)$$

(observe that, by hypothesis, $\lambda_i = j\alpha_i$ where α_i is a nonzero real number).

To prove the result, we need to show that there is a choice of k and

$$n(s) = \prod_{i=1}^{q-1}(s - \sigma_i) ,$$

such that the roots of the polynomial

$$P(s) = d(s)D_\mu(s) - kn(s)N_\mu(s)$$

are in $\mathbb{C}^-$ for each value of $\mu \in \mathcal{P}^*$.

The proof of this reposes on a simple root locus argument. More specifically, we show that, if the roots of $n(s)$ and the sign of k are appropriately chosen, the q roots of $P(s)$ which are on the imaginary axis at $k = 0$ move into $\mathbb{C}^-$ for small $|k|$. Since the other $\tilde{n}$ roots of $P(s)$ remain close to those of $D_\mu(s)$, which are in $\mathbb{C}^-$ for all $\mu \in \mathcal{P}^*$, and $\mathcal{P}^*$ is compact, this proves the claim.

Suppose none of the roots of $n(s)$ is on the real axis. Then, the root of $P(s)$ which is at $s = 0$ for $k = 0$ moves into $\mathbb{C}^-$ if

$$\mathrm{sgn}(k) \neq \mathrm{sgn}(T_\mu(0)) \; .$$

Thanks to the hypothesis (4.11), the sign of $T_\mu(0)$ is always the same for all $\mu \in \mathcal{P}^*$. Thus, we can choose the sign of k so that the root in question moves in $\mathbb{C}^-$ for all $\mu \in \mathcal{P}^*$.

The other roots of $P(s)$ which are on the imaginary axis for $k = 0$ move into $\mathbb{C}^-$ or into $\mathbb{C}^+$ depending on angle of departure of the root locus at each one of these (simple) roots. Now, observe that the angle of departure $\alpha_j(\mu)$ of the root locus at λ_j is given by

$$\begin{aligned}\alpha_j(\mu) = \mathrm{Arg}(k) + \mathrm{Arg}(T_\mu(\lambda_j)) - \textstyle\sum_{\substack{i=1\\ i\neq j}}^{q-1} \mathrm{Arg}(\lambda_j - \lambda_i) - \mathrm{Arg}(\lambda_j) \\ + \textstyle\sum_{i=1}^{q-1} \mathrm{Arg}(\lambda_j - \sigma_i) \qquad (\mathrm{mod}\ 2\pi).\end{aligned}$$

It is easy to realize that there exist σ_i's such that

$$\begin{aligned}\mathrm{Arg}(k) + \mathrm{Arg}(T_0(\lambda_j)) - \textstyle\sum_{\substack{i=1\\ i\neq j}}^{q-1} \mathrm{Arg}(\lambda_j - \lambda_i) - \mathrm{Arg}(\lambda_j) \\ + \textstyle\sum_{i=1}^{q-1} \mathrm{Arg}(\lambda_j - \sigma_i) = \pi \qquad (\mathrm{mod}\ 2\pi),\end{aligned} \tag{4.13}$$

for $j = 1, \ldots, q-1$. In fact, it suffices to set $\sigma_i = \lambda_i - \varepsilon e^{j\theta_i}$; observe that this yields

$$\sum_{i=1}^{q-1} \mathrm{Arg}(\lambda_j - \sigma_i) = \theta_j + \psi_j(\varepsilon, \theta_1, \ldots, \theta_{q-1})$$

where $\psi_j(\varepsilon, \theta_1, \ldots, \theta_{q-1})$ is a quantity which is independent of $\theta_1, \ldots,$ θ_{q-1} at $\varepsilon = 0$, and use the implicit function theorem to determine that for small ε the equations (4.13) have solutions $\sigma_1, \ldots, \sigma_{q-1}$ none of which is a real number.

With the σ_i's chosen to satisfy (4.13), we obtain

$$\alpha_j(\mu) = \pi + \mathrm{Arg}\Big(\frac{T_\mu(\lambda_j)}{T_0(\lambda_j)}\Big) \qquad (\mathrm{mod}\ 2\pi).$$

Then, thanks to the hypothesis (4.11), for all $\mu \in \mathcal{P}^*$ the angle of departure of the root locus at λ_j satisfies

$$\frac{\pi}{2} < \alpha_j(\mu) < \frac{3\pi}{2} \; .$$

Suppose, without loss of generality, $\text{sgn}(T_0(0)) > 0$. The previous arguments show that all roots of $P(s)$ which are on the imaginary axis for $k = 0$ move into $\mathbb{C}^-$ for $k < 0$, so long as $|k|$ is small. Since $\mathcal{P}^*$ is compact, there is a number $k^* < 0$ such that the roots of $P(s)$ are in $\mathbb{C}^-$ for all $\mu \in \mathcal{P}^*$, when $k = k^*$. Choosing $b_0, \ldots, b_{q-1}$ so that

$$b_0 + b_1 s + \ldots + b_{q-1} s^{q-1} = k^* n(s)$$

completes the proof. ◁

4.3 The special case of systems in triangular form

We describe in this section some special classes of single-input single-output nonlinear system for which the various conditions for structurally stable local regulation described in the previous sections can easily be tested. Consider, for example, systems which can be modeled by equations of the following form

$$\begin{aligned}
\dot{x}_1 &= a_2(\mu)x_2 + p_1(x_1, w, \mu) \\
\dot{x}_2 &= a_3(\mu)x_3 + p_2(x_1, x_2, w, \mu) \\
&\cdots \\
\dot{x}_{n-1} &= a_n(\mu)x_n + p_{n-1}(x_1, x_2, \ldots, x_{n-1}, w, \mu) \\
\dot{x}_n &= p_n(x_1, x_2, \ldots, x_n, w, \mu) + b(\mu)u \\
e &= c(\mu)x_1 - q(w, \mu) \, ,
\end{aligned} \tag{4.14}$$

in which, for each $\mu \in \mathcal{P}^*$, the coefficients $a_2(\mu)$, $a_3(\mu)$, $\ldots$, $a_n(\mu)$, $b(\mu)$, $c(\mu)$ are nonzero, $q(0, \mu) = 0$ and $p_i(0, \ldots, 0, 0, \mu) = 0$ for $1 \leq i \leq n$. The system in question is a system of the form

$$\begin{aligned}
\dot{x} &= F(\mu)x + G(\mu)u + P(x, w, \mu) \\
e &= H(\mu)x - q(w, \mu)
\end{aligned} \tag{4.15}$$

with

$$\begin{aligned}
F(\mu) &= \begin{pmatrix} 0 & a_2(\mu) & 0 & \cdots & 0 \\ 0 & 0 & a_3(\mu) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_n(\mu) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad G(\mu) = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ b(\mu) \end{pmatrix} \\
H(\mu) &= \begin{pmatrix} c(\mu) & 0 & 0 & \cdots & 0 \end{pmatrix} ,
\end{aligned}$$

and the vector $P(x, w, \mu)$ exhibits a triangular dependence of the individual components of x. In particular, its relative degree is equal to n, for all μ.

For this system it is immediate to find a solution of the regulator equations (4.4), which is defined for all w and all μ in $\mathcal{P}^*$. In fact, observe that the second equation of (4.4), i.e.

$$h(\pi^a(w, \mu), w, \mu) = 0 ,$$

directly provides the expression of the first component $\pi_1^a(w, \mu)$ of $\pi^a(w, \mu)$, which is

$$\pi_1^a(w, \mu) = \frac{q(w, \mu)}{c(\mu)} .$$

Because of the special structure of system (4.14), the first equation of (4.4) uniquely determines all other components of $\pi^a(w, \mu)$. In particular, its second component $\pi_2^a(w, \mu)$ is determined by the identity

$$\frac{\partial \pi_1^a}{\partial w} s(w) = a_2(\mu)\pi_2^a(w, \mu) + p_1(\pi_1^a(w, \mu), w, \mu) ,$$

which yields

$$\pi_2^a(w, \mu) = \frac{1}{a_2(\mu)}\Big(\frac{\partial \pi_1^a}{\partial w} s(w) - p_1(\pi_1^a(w, \mu), w, \mu))\Big) .$$

This procedure can be iterated up to the last equation, which eventually provides the unique expression for $c^a(w, \mu)$, which is

$$c^a(w, \mu) = \frac{1}{b(\mu)}\Big(\frac{\partial \pi_n^a}{\partial w} s(w) - p_n(\pi_1^a(w, \mu), \ldots, \pi_n^a(w, \mu), w, \mu))\Big) . \tag{4.16}$$

Observe also that the matrices $A(\mu), B(\mu), C(\mu)$ which characterize the linear approximation of (4.15) at the equilibrium $(x, w) = (0, 0)$ have the following structure

$$A(\mu) = F(\mu) + \Big[\frac{\partial P(x, w, \mu)}{\partial x}\Big]_{(0,0,\mu)}, \quad B(\mu) = G(\mu), \quad C(\mu) = H(\mu) ,$$

where

$$\Big[\frac{\partial P(x, w, \mu)}{\partial x}\Big]_{0,0,\mu}$$

is a lower triangular matrix. Thus, in view of the special structure of $F(\mu), G(\mu), H(\mu)$, for all $\mu \in \mathcal{P}^*$ the pair $(A(\mu), B(\mu))$ is controllable, the pair $(C(\mu), A(\mu))$ is stabilizable and the matrix

$$\begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix}$$

is nonsingular for all λ.

If the (unique) expression for $c^{\mathrm{a}}(w, \mu)$ provided by (4.16) is such that, for some integer p and some function $\phi(\xi_1, \xi_2, \dots, \xi_q)$,

$$L_s^q c^{\mathrm{a}}(w, \mu) = \phi(c^{\mathrm{a}}(w, \mu), L_s c^{\mathrm{a}}(w, \mu), \dots, L_s^{q-1} c^{\mathrm{a}}(w, \mu)) \,, \qquad (4.17)$$

then, in view of the results presented in Chapter 2, the autonomous system with output $\{\mathbb{R}^r \times \mathcal{P}^*, s^{\mathrm{a}}, c^{\mathrm{a}}\}$ is immersed into a nonlinear system $\{\mathbb{R}^q, \varphi, \gamma\}$ in which

$$\varphi(\xi) = \begin{pmatrix} \xi_2 \\ \xi_3 \\ \cdots \\ \xi_q \\ \phi(\xi_1, \xi_2, \cdots, \xi_q) \end{pmatrix}, \qquad \gamma(\xi) = \xi_1 \,.$$

If this is the case, all the conditions for the existence of a controller solving the problem of structurally stable local output regulation described in section 4.1 are fulfilled.

A special case in which a condition of the form (4.17) holds is when the exosystem is a linear system and, for any $\mu \in \mathcal{P}^*$, $c^{\mathrm{a}}(w, \mu)$ is a polynomial in w whose degree does not exceed a fixed number independent of μ. For the special system (4.14), this turns out to be the case whenever, for each $\mu \in \mathcal{P}^*$, the $p_i(x_1, x_2, \dots, x_i, w, \mu)$'s are polynomials in x_1, x_2, $\dots$, x_i, w, and $q(w, \mu)$ is a polynomial in w, of degree not exceeding a fixed number independent of μ.

In fact, the recursive solution of the equations (4.4) presented above yields, at each step, a component of $\pi^{\mathrm{a}}(w, \mu)$ which is a polynomial in w, and whose degree does not exceed a fixed number independent of μ. Thus, eventually, also the unique expression found for $c^{\mathrm{a}}(w, \mu)$ is a polynomial in w, whose degree does not exceed a fixed number. This being the case, on the basis of the results presented in Chapter 2, it is immediate to conclude that the autonomous system with output $\{\mathbb{R}^r \times \mathcal{P}^*, s^{\mathrm{a}}, c^{\mathrm{a}}\}$ is immersed into a linear observable system.

These considerations can be extended to the more general class of systems modeled by equations of the form

$$\begin{array}{rcl} \dot{z} & = & Z(\mu)z + p_0(x_1, w, \mu) \\ \dot{x}_1 & = & a_2(\mu)x_2 + p_1(z, x_1, w, \mu) \\ \dot{x}_2 & = & a_3(\mu)x_3 + p_2(z, x_1, x_2, w, \mu) \\ & & \cdots \\ \dot{x}_{n-1} & = & a_n(\mu)x_n + p_{n-1}(z, x_1, x_2, \ldots, x_{n-1}, w, \mu) \\ \dot{x}_n & = & p_n(z, x_1, x_2, \ldots, x_n, w, \mu) + b(\mu)u \\ e & = & c(\mu)x_1 - q(w, \mu) \end{array} \tag{4.18}$$

in which $z \in \mathbb{R}^m$, provided that, for each $\mu \in \mathcal{P}^*$, $p_0(x_1, w, \mu)$, $p_1(z, x_1, w, \mu)$, $\ldots$ $p_n(z, x_1, x_2, \ldots, x_n, w, \mu)$ and $q(w, \mu)$ are polynomials in z,x_1, x_2, $\ldots$, x_i, w, of degree not exceeding a fixed number independent of μ, and the eigenvalues of the matrix $Z(\mu)$ have nonzero real part.

Also this case, in fact, if the exosystem is a neutrally stable linear system, the regulator equations (4.4) have a unique globally defined solution, in which $c^{a}(w, \mu)$ is a polynomial in w, whose degree does not exceed a fixed number. To see why this is the case, split $\pi^{a}(w, \mu)$ as

$$\begin{pmatrix} \zeta(w, \mu) \\ \pi_1^{a}(w, \mu) \\ \cdots \\ \pi_n^{a}(w, \mu) \end{pmatrix} ,$$

in which $\zeta(w, \mu)$ and the $\pi_i^{a}(w, \mu)$'s correspond to the partition of the state vector of (4.18) into z and the x_i's respectively.

As in the previous case, in fact, observe that the second equation of (4.4) directly provides the expression of $\pi_1^{a}(w, \mu)$, namely

$$\pi_1^{a}(w, \mu) = \frac{q(w, \mu)}{c(\mu)} .$$

Then, observe that the equation for $\zeta(w, \mu)$ is an equation of the form

$$\frac{\partial \zeta}{\partial w} S w = Z(\mu)\zeta(w, \mu) + p_0(\pi_1^{a}(w, \mu), w, \mu) \tag{4.19}$$

where $p_0(\pi_1^{a}(w, \mu), w, \mu)$ is a polynomial in w, whose degree does not exceed a fixed number (independent of μ). Let p denote this number. For each fixed μ, the equation in question is precisely an equation of the form (1.20), discussed in section 1.2. Therefore, by Lemma 1.2, this equation has a unique solution $\zeta(w,\mu)$ whose entries are polynomials in w, whose degree does not exceed the number p.

Once the expression of $\zeta(w,\mu)$ has been determined, the same recursive method of solution described in the previous case can be used, to yield all remaining components $\pi_2^{\mathrm{a}}(w,\mu)$, ... , $\pi_n^{\mathrm{a}}(w,\mu)$ and $c^{\mathrm{a}}(w,\mu)$. Moreover, since $\zeta(w,\mu)$ is a polynomial in w, it is easily seen that all the $\pi_2^{\mathrm{a}}(w,\mu)$'s as well as $c^{\mathrm{a}}(w,\mu)$ are polynomials in w, whose degree does not exceed a fixed number independent of μ. As a consequence, the unique solution $(\pi_2^{\mathrm{a}}(w,\mu), c^{\mathrm{a}}(w,\mu))$ of the regulator equations (4.4) is such that a condition of the form (4.17) holds (actually, for some *linear* function $\phi(\xi_1,\xi_2,\ldots,\xi_p)$).

4.4 A globally defined error-zeroing invariant manifold

Motivated by the analysis presented in the previous section, we consider henceforth output regulation problems for systems modeled by equations of the form

$$\begin{aligned} \dot z &= Z(\mu)z + p_0(x_1,w,\mu) \\ \dot x &= F(\mu)x + G(\mu)u + P(z,x,w,\mu) \\ e &= H(\mu)x - q(w,\mu) \end{aligned} \tag{4.20}$$

where

$$F(\mu) = \begin{pmatrix} 0 & a_2(\mu) & 0 & \cdots & 0 \\ 0 & 0 & a_3(\mu) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_n(\mu) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad G(\mu) = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ b(\mu) \end{pmatrix}$$

$$H(\mu) = \begin{pmatrix} c(\mu) & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$P(z,x,w,\mu) = \begin{pmatrix} p_1(z,x_1,w,\mu) \\ p_2(z,x_1,x_2,w,\mu) \\ \cdots \\ p_{n-1}(z,x_1,x_2,\ldots,x_{n-1},w,\mu) \\ p_n(z,x_1,x_2,\ldots,x_n,w,\mu) \end{pmatrix}.$$

In what follows its is assumed that, for each $\mu \in \mathcal{P}$, $q(w,\mu)$, $p_0(x_1,w,\mu)$ $p_1(z,x_1,w,\mu)$, ..., $p_n(z,x_1,x_2,\ldots,x_n,w,\mu)$ are polynomials in z, x_1, x_2, ... , x_n, w of degree not exceeding a fixed number

k, independent of μ. Moreover, it is assumed that, for all $\mu \in \mathcal{P}$, the eigenvalues of $Z(\mu)$ have negative real part. Finally, it is also assumed (without loss of generality if the compact set $\mathcal{P}$ is connected) that $a_2(\mu) > 0$, $a_3(\mu) > 0$, ... , $a_n(\mu) > 0$, $b(\mu) > 0$ and $c(\mu) > 0$ for all $\mu \in \mathcal{P}$.

Under these hypotheses, the equations (4.4) have a unique and globally defined solution $\pi^a(w,\mu), c^a(w,\mu)$ in which $c^a(w,\mu)$ satisfies an identity of the form.

$$L_s^q c^a(w,\mu) = a_0 c^a(w,\mu) - a_1 L_s c^a(w,\mu) - \ldots - a_{q-1} L_s^{q-1} c^a(w,\mu) \; .$$

This, in turn, uniquely determines the existence of a $q \times q$ matrix Φ, a $1 \times q$ row vector Γ, and a globally defined mapping $\tau(w,\mu)$ such that

$$\begin{aligned} \frac{\partial \tau^a(w,\mu)}{\partial w} S w &= \Phi \tau^a(w,\mu) \\ c^a(w,\mu) &= \Gamma \tau^a(w,\mu) \; . \end{aligned} \tag{4.21}$$

As a matter of fact, this occurs for

$$\tau^a(w,\mu) = \begin{pmatrix} c^a(w,\mu) \\ L_s c^a(w,\mu) \\ \cdot \\ L_s^{q-2} c^a(w,\mu) \\ L_s^{q-1} c^a(w,\mu) \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{q-1} \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} .$$

Consider now a feedback law of the form

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + Le \\ \dot{\xi}_1 &= \Phi \xi_1 + Ne \\ u &= \alpha(\xi_0) + T\xi_1 \; . \end{aligned} \tag{4.22}$$

in which $\alpha(\xi_0)$ is a (possibly nonlinear) function, smooth in a neighborhood of $\xi_0 = 0$ and such that $\alpha(0) = 0$.

Then, it is possible to prove that the following property holds.

Proposition 4.4 *Suppose (4.22) asymptotically stabilizes the linear approximation of (4.20) at the equilibrium point* $(\xi_0, \xi_1, z, x) = (0,0,0,0)$, $(w,\mu) = (0,0)$. *Suppose* $\sigma(K) \in \mathbb{C}^-$. *Then, there exists a* $q \times q$ *matrix* Π *satisfying*

$$\begin{aligned} \Phi\Pi &= \Pi\Phi \\ T\Pi &= \Gamma , \end{aligned} \tag{4.23}$$

where Φ *and* Γ *are defined as in (4.21). As a consequence, the composite system*

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + L(H(\mu)x - q(w,\mu)) \\ \dot{\xi}_1 &= \Phi\xi_1 + N(H(\mu)x - q(w,\mu)) \\ \dot{z} &= Z(\mu)z + p_0(x_1, w, \mu) \\ \dot{x} &= F(\mu)x + G(\mu)(\alpha(\xi_0) + T\xi_1) + P(z,x,w,\mu) \\ \dot{w} &= Sw \end{aligned} \tag{4.24}$$

has a globally defined center manifold

$$\begin{aligned} \mathcal{M}_c = \{(\xi_0, \xi_1, z, x, w) :\\ \xi_0 = 0, \xi_1 = \Pi\tau^a(w,\mu), z = \zeta(w,\mu), x = \pi^a(w,\mu)\} \end{aligned}$$

at $(\xi_0, \xi_1, z, x, w) = (0,0,0,0,0)$.

Proof. Note that the linear approximation of (4.24) at $(\xi_0, \xi_1, z, x, w) = (0,0,0,0,0,0)$, is a system of the form

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + LH(\mu)x + LQ(\mu)w \\ \dot{\xi}_1 &= \Phi\xi_1 + NH(\mu)x + NQ(\mu)w \\ \dot{z} &= Z(\mu)z + T_x(\mu)H(\mu)x + T_w(\mu)w \\ \dot{x} &= F(\mu)x + G(\mu)(M\xi_0 + T\xi_1) + P_z(\mu)z + P_x(\mu)x + P_w(\mu)w \\ \dot{w} &= Sw \, . \end{aligned} \tag{4.25}$$

By hypothesis, the matrix

$$\begin{pmatrix} K & 0 & 0 & LH(0) \\ 0 & \Phi & 0 & NH(0) \\ 0 & 0 & Z(0) & T_x(0)H(0) \\ G(0)M & G(0)T & P_z(0) & F(0) + P_x(0) \end{pmatrix} \tag{4.26}$$

has all eigenvalues with negative real part. Since the matrix Φ has all eigenvalues on the imaginary axis, the Sylvester equation

$$\begin{pmatrix} K & 0 & 0 & LH(0) \\ 0 & \Phi & 0 & NH(0) \\ 0 & 0 & Z(0) & T_z(0)H(0) \\ G(0)M & G(0)T & P_z(0) & F(0)+P_x(0) \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_z \\ X_x \end{pmatrix} - \begin{pmatrix} X_0 \\ X_1 \\ X_z \\ X_x \end{pmatrix} \Phi = \begin{pmatrix} 0 \\ Y \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution for any choice of Y. Thus, in particular, for any Y there exist X_1 and X_x such that

$$\Phi X_1 + NH(0)X_x - X_1\Phi = Y \, .$$

From this, using the results illustrated in section 5.1, it is immediate to conclude that an identity of the form

$$\Phi X_1 + NH(0)X_x - X_1\Phi = 0$$

necessarily implies

$$\Phi X_1 - X_1\Phi = 0 \qquad \text{and} \qquad HX_x = 0 \, .$$

Using again the hypotheses that the matrix (4.26) has all eigenvalues with negative real part and the matrix Φ has all eigenvalues on the imaginary axis, note that the Sylvester equation

$$\begin{pmatrix} K & 0 & 0 & LH(0) \\ 0 & \Phi & 0 & NH(0) \\ 0 & 0 & Z(0) & T_z(0)H(0) \\ G(0)M & G(0)T & P_z(0) & F(0)+P_x(0) \end{pmatrix} \begin{pmatrix} X_0 \\ \Pi \\ X_z \\ X_x \end{pmatrix} - \begin{pmatrix} X_0 \\ \Pi \\ X_z \\ X_x \end{pmatrix} \Phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ G(0)\Gamma \end{pmatrix}$$

has a unique solution for any choice of Γ (thus, in particular, for the Γ defined in (4.21)). This equation can be split as follows

$$KX_0 + LH(0)X_x - X_0\Phi = 0 \tag{4.27}$$

$$\Phi\Pi + NH(0)X_x - \Pi\Phi = 0 \tag{4.28}$$

$$Z(0)X_z + T_x(0)H(0)X_x - X_z\Phi = \tag{4.29}$$

$$G(0)MX_0 + G(0)T\Pi + P_z(0)X_z + (F(0) + P_x(0))X_x - X_x\Phi = G(0)\Gamma \tag{4.30}$$

In view of the previous discussion, the second of these identities implies

$$H(0)X_x = 0 \tag{4.31}$$

and, of course

$$\Phi\Pi = \Pi\Phi \, ,$$

which is one of the two identities it is required to prove. Replacing (4.31) into (4.27) yields

$$KX_0 - X_0\Phi = 0$$

which in turn, since K has eigenvalues with negative real part by hypothesis, yields

$$X_0 = 0 \, . \tag{4.32}$$

Using (4.31) in (4.29) yields

$$Z(0)X_z - X_z\Phi = 0$$

which again implies

$$X_z = 0 \tag{4.33}$$

since $Z(0)$ has no eigenvalue with zero real part. Finally, replacing (4.32) and (4.33) into (4.30) yields

$$G(0)T\Pi + (F(0) + P_x(0))X_x - X_x\Phi = G(0)\Gamma \, . \tag{4.34}$$

Using the identity (4.31), which expresses the fact that the first row of X_x is zero, and keeping in mind the special structures of $F(0), G(0)$ and $P_x(0)$ (which is lower triangular), the relation (4.34) yields

$$X_x = 0$$

i.e.

$$G(0)T\Pi = G(0)\Gamma \, .$$

which concludes the proof of (4.23).

The fact that the manifold $\mathcal{M}_c$ is invariant can be checked by direct substitution. ◃

. Noting that the error e is zero on $\mathcal{M}_c$, the issue is now to choose K, L, N, T, $\alpha(\xi_0)$ in (4.22) so that $\mathcal{M}_c$ is semiglobally attractive. This issue will be addressed in the next section.

4.5 Semiglobal robust regulation

Consider again system (4.20), driven by the controller (4.22) and subject to a disturbance w generated by the exosystem

$$\dot{w} = Sw ,$$

i.e. system (4.24). Suppose the hypotheses of Proposition 4.4 hold and consider the following (globally defined) change of coordinates in the state space of (4.24)

$$\begin{aligned} \tilde{\xi}_1 &= \xi_1 - \Pi\tau^{\mathrm{a}}(w,\mu) \\ \tilde{z} &= z - \zeta(w,\mu) \\ \tilde{x} &= x - \pi^{\mathrm{a}}(w,\mu) . \end{aligned} \tag{4.35}$$

In these new coordinates, system (4.24) is represented by equations of the form

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + Lc(\mu)\tilde{x}_1 \\ \dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + Nc(\mu)\tilde{x}_1 \\ \dot{\tilde{z}} &= Z(\mu)\tilde{z} + \tilde{p}_0(\tilde{x}_1, w, \mu) \\ \dot{\tilde{x}} &= F(\mu)\tilde{x} + G(\mu)(\alpha(\xi_0) + T\tilde{\xi}_1) + \tilde{P}(\tilde{z}, \tilde{x}, w, \mu) \\ \dot{w} &= Sw , \end{aligned} \tag{4.36}$$

and the manifold $\mathcal{M}_c$ is the subset where

$$\xi_0 = 0, \qquad \tilde{\xi}_1 = 0, \qquad \tilde{z} = 0, \qquad \tilde{x} = 0 .$$

Thus, since this manifold is invariant,

$$\begin{aligned} \tilde{p}_0(0, w, \mu) &= 0 \\ \tilde{P}(0, 0, w, \mu) &= 0 \end{aligned}$$

for every $(w, \mu) \in \mathbb{R}^r \times \mathcal{P}$. Moreover, it is easy to check that $\tilde{P}(\tilde{z}, \tilde{x}, w, \mu)$ has the following structure

$$\tilde{P}(\tilde{z}, \tilde{x}, w, \mu) = \begin{pmatrix} \tilde{p}_1(\tilde{z}, \tilde{x}_1, w, \mu) \\ p_2(\tilde{z}, \tilde{x}_1, \tilde{x}_2, w, \mu) \\ \cdots \\ p_{n-1}(\tilde{z}, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}, w, \mu) \\ p_n(\tilde{z}, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, w, \mu) \end{pmatrix} . \tag{4.37}$$

By construction the tracking error

$$e = \tilde{x}_1$$

is zero on $\mathcal{M}_c$ and if $\mathcal{M}_c$ is attractive, i.e.

$$\lim_{t\to\infty} \xi_0(t) = 0, \quad \lim_{t\to\infty} \tilde{\xi}_1(t) = 0, \quad \lim_{t\to\infty} \tilde{z}(t) = 0, \quad \lim_{t\to\infty} \tilde{x}(t) = 0 ,$$

then clearly also

$$\lim_{t\to\infty} e(t) = 0 .$$

Observe now that, if we let w° denote the value at time $t = 0$ of the state of the exosystem, system (4.36) can be rewritten in the form of a time-varying system as

$$\begin{aligned}
\dot{\xi}_0 &= K\xi_0 + Lc(\mu)\tilde{x}_1 \\
\dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + Nc(\mu)\tilde{x}_1 \\
\dot{\tilde{z}} &= Z(\mu)\tilde{z} + \tilde{p}_0(\tilde{x}_1, exp(St)w^\circ, \mu) \\
\dot{\tilde{x}} &= F(\mu)\tilde{x} + G(\mu)(\alpha(\xi_0) + T\tilde{\xi}_1) + \tilde{P}(\tilde{z}, \tilde{x}, exp(St)w^\circ, \mu) .
\end{aligned} \tag{4.38}$$

The problem to be addressed is to find matrices K, L, N, T and a function $\alpha(\xi_0)$ such that the equilibrium $(\xi_0, \tilde{\xi}_1, \tilde{z}, \tilde{x}) = (0, 0, 0, 0)$ of this system is asymptotically stable, with a basin of attraction containing a fixed compact set of initial states, robustly in w° and μ. Suppose the matrix N has been fixed (as we will see in a moment, it suffices to choose for N any matrix such that the pair (Φ, N) is controllable). Then the problem in question can be viewed as the problem of determining a dynamic feedback law of the form

$$\begin{aligned}
\dot{\xi}_0 &= K\xi_0 + Le \\
u &= \alpha(\xi_0) + T\tilde{\xi}_1 .
\end{aligned} \tag{4.39}$$

which robustly asymptotically stabilizes (with a basin of attraction containing a fixed compact set of initial states) an uncertain time-varying nonlinear system modeled by equations of the form

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + Nc(\mu)\tilde{x}_1 \\
\dot{\tilde{z}} &= Z(\mu)\tilde{z} + \tilde{p}_0(c(\mu)\tilde{x}_1, exp(St)w^\circ, \mu) \\
\dot{\tilde{x}} &= F(\mu)\tilde{x} + G(\mu)u + \tilde{P}(\tilde{z}, \tilde{x}, exp(St)w^\circ, \mu) \\
e &= c(\mu)\tilde{x}_1 .
\end{aligned} \tag{4.40}$$

System (4.40) is an uncertain system because the actual value of μ as well as that of the initial state w° of the exosystem are not known. For consistency with the hypothesis on μ, we assume that the initial state w° of the exosystem ranges over an a priori known compact set

$\mathcal{W} \in \mathbb{R}^r$. In other words, system (4.40) can be regarded as a system in which the terms $\tilde{p}_0(c(\mu)\tilde{x}_1, exp(St)w^\circ, \mu)$ and $\tilde{P}(\tilde{z}, \tilde{x}, exp(St)w^\circ, \mu)$ are bounded (recall that the exosystem is assumed to be neutrally stable) functions of t, vanishing at $(\tilde{z}, \tilde{x}) = (0, 0)$ for all $t \in \mathbb{R}$ and for all values of (w°, μ), unknown parameters ranging over a fixed compact set $\mathcal{W} \times \mathcal{P}$.

In order to explain how robust asymptotic stability can be obtained by means of a feedback of the form (4.39), it is useful to describe first how a system of the (4.40) can be robustly stabilized (in a semiglobal sense) by means of a special kind of *memoryless and linear* state-feedback.

To this end we begin by showing how system (4.40) can be given a simpler structure by replacing the actual state variables $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ by the error variable e and its first $n-1$ derivatives with respect to time. Define

$$\eta_1 = \eta_1(\tilde{z}, \tilde{x}, t) = c(\mu)\tilde{x}_1$$

and, recursively, for $2 \le i \le n$,

$$\eta_i = \eta_i(\tilde{z}, \tilde{x}, t) = \frac{\partial \eta_{i-1}}{\partial \tilde{z}}\dot{\tilde{z}} + \frac{\partial \eta_{i-1}}{\partial \tilde{x}}\dot{\tilde{x}} + \frac{\partial \eta_{i-1}}{\partial t},$$

and observe that, because of the special structure of the (4.37), for all $1 \le i \le n$, η_i can be given an expression of the form

$$\eta_i = k_i(\mu)\tilde{x}_i + \phi_i(\tilde{z}, \tilde{x}_1, \ldots, \tilde{x}_{i-1}, exp(St)w^\circ, \mu)$$

in which $k_i(\mu)$ is nowhere zero on $\mathcal{P}$. Thus, the mapping

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \cdots \\ \tilde{x}_n \end{pmatrix} \mapsto \begin{pmatrix} \eta_1(\tilde{z}, \tilde{x}, t) \\ \eta_2(\tilde{z}, \tilde{x}, t) \\ \cdots \\ \eta_n(\tilde{z}, \tilde{x}, t) \end{pmatrix} \tag{4.41}$$

can be used to globally define a (partial) change of coordinates in the state space of (4.40). Observe that, by construction,

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \cdots \\ \eta_n(t) \end{pmatrix} = \begin{pmatrix} e(t) \\ e^{(1)}(t) \\ \cdots \\ e^{(n-1)}(t) \end{pmatrix}.$$

Remark. Note that neither the coordinates $\tilde{x}_1, \ldots \tilde{x}_n$, nor the coordinates $\eta_1, \ldots, \eta_n$ will ever be actually available for feedback, because they depend on the unknown parameters w° and μ. However this apparent inconvenience can be removed if, as it will be shown later on, reasonable asymptotic estimates the first $n-1$ derivatives of the error e can be generated. ◁

Having changed the coordinates in this way, the control system (4.40) reduces to a system of the form

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + N\eta_1 \\
\dot{\tilde{z}} &= Z(\theta)\tilde{z} + p(\eta_1, t, \theta) \\
\dot{\eta}_1 &= \eta_2 \\
\dot{\eta}_2 &= \eta_3 \\
&\cdots \\
\dot{\eta}_{n-1} &= \eta_n \\
\dot{\eta}_n &= k(\theta)(u + q(\tilde{z}, \eta, t, \theta)) \ ,
\end{aligned}$$

in which $p(\eta_1, t, \theta)$ and $q(\tilde{z}, \eta, t, \theta)$ are bounded functions of t, vanishing at $(\tilde{z}, \eta) = (0, 0)$ for all t and θ, and in which $\theta = (w^\circ, \mu)$ is a $(r+p)$-tuple of unknown parameters, ranging over a fixed compact set Θ. Of course, $Z(\theta) = Z(\mu)$ in this notation and $k(\theta)$ in nowhere zero. This system can be put in more compact form as

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + NH\eta \\
\dot{\tilde{z}} &= Z(\theta)\tilde{z} + p(H\eta, t, \theta) \\
\dot{\eta} &= F\eta + Gk(\theta)(u + q(\tilde{z}, \eta, t, \theta)) \ ,
\end{aligned} \tag{4.42}$$

where

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix} \tag{4.43}$$
$$H = (1 \quad 0 \quad \cdots \quad 0) \ .$$

Note also that, setting

$$x_1 = \tilde{z}, \qquad x_2 = \begin{pmatrix} \tilde{\xi}_1 \\ \eta \end{pmatrix}, \qquad A = \begin{pmatrix} \Phi & NH \\ 0 & F \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ G \end{pmatrix}$$

the system under consideration can be further simplified to

$$\begin{aligned}
\dot{x}_1 &= Z(\theta)x_1 + p(HEx_2, t, \theta) \\
\dot{x}_2 &= Ax_2 + Bk(\theta)(u + q(x_1, Ex_2, t, \theta)) \ ,
\end{aligned} \tag{4.44}$$

where

$$E = (0 \quad I_n) \ .$$

Note that, if N is such that (Φ, N) is a controllable pair, also the pair (A, B) is controllable. Moreover, since all the entries on the last row of F are zero and the only nonzero entry in G is the last one (equal to 1), the vector x_2 and the matrices A and B can also be re-partitioned as

$$x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\dim(x_{21}) = q + n - 1$ and $\dim(x_{22}) = 1$. It is easy to check that, by construction, (A_{11}, A_{12}) is a controllable pair.

The design procedure which will be illustrated relies upon the following hypothesis on (4.44).

Assumption A. There exists a positive definite proper smooth function $V_1(x_1)$ satisfying

$$\frac{\partial V_1}{\partial x_1}(Z(\theta)x_1 + p(v,t,\theta)) \le -\alpha(V_1(x_1)) + c|v|^2 \tag{4.45}$$

for all x_1, v, t and all $\theta \in \Theta$, where $\alpha(r)$ is a $\mathcal{K}_\infty$ function, which is also assumed to be continuously differentiable, and $c > 0$.

Moreover,

$$|q(x_1, Ex_2, t, \theta)|^2 \le \alpha(V_1(x_1)) + \gamma(|x_2|^2) \tag{4.46}$$

for all x_1, x_2, t and all $\theta \in \Theta$, where $\gamma(r)$ is a $\mathcal{K}_\infty$ function, which is also assumed to be continuously differentiable near $r = 0$.$\triangleleft$

Using this hypothesis, the following result can be proven.

Lemma 4.5 *Choose any matrix K_1 such that the eigenvalues of $A_{11} + A_{12}K_1$ have negative real part, and let P be the unique (positive definite) solution of the Lyapunov equation*

$$(A_{11} + A_{12}K_1)^{\mathrm{T}}P + P(A_{11} + A_{12}K_1) = -I \ .$$

Consider the positive definite function

$$V_2(x_2) = x_{21}^{\mathrm{T}} P x_{21} + (x_{22} - K_1 x_{21})^2 \ .$$

Suppose estimate (4.46) holds. Then, for any (arbitrarily small) number $b > 0$ and any (arbitrarily large) number $d > 0$ there is a matrix K_2 such that

$$\frac{\partial V_2}{\partial x_2}[Ax_2 + Bk(\theta)(K_2x_2 + g(x_1, Ex_2, t, \theta))] \leq -aV_2(x_2) + b\alpha(V_1(x_1)) \tag{4.47}$$

for all x_1, for all t, for all $\theta \in \Theta$ and for all x_2 such that $|x_2| \leq d$, where $a > 0$ is a number depending only on $V_2(x_2)$ (and not on b and d).

Proof. For convenience, set

$$\begin{aligned} Lx_2 &= 2(x_{22} - K_1x_{21}) \\ Kx_2 &= -K_1A_{11}x_{21} - K_1A_{21}x_{22} + x_{21}^{\mathrm{T}}PA_{21} \\ g(x,t,\theta) &= g(x_1, Ex_2, t, \theta)\ , \end{aligned}$$

and compute the derivative $\dot{V}_2(x_2)$ of $V_2(x_2)$ along the trajectories of (4.44), to obtain

$$\begin{aligned} &\dot{V}_2(x_2) \\ &= 2x_{21}^{\mathrm{T}}P(A_{11}x_{21} + A_{21}x_{22}) + 2(x_{22} - K_1x_{21})k(\theta)[u + g(x,t,\theta)] \\ &\quad -2(x_{22} - K_1x_{21})K_1(A_{11}x_{21} + A_{21}x_{22}) \\ &\quad -2x_{21}^{\mathrm{T}}PA_{21}(x_{22} - K_1x_{21}) + 2x_{21}^{\mathrm{T}}PA_{21}(x_{22} - K_1x_{21}) \\ &= 2x_{21}^{\mathrm{T}}P(A_{11} + A_{21}K_1)x_{21} + Lx_2k(\theta)[u + g(x,t,\theta) + \frac{1}{k(\theta)}Kx_2] \\ &= -|x_{21}|^2 + Lx_2k(\theta)[u + g(x,t,\theta) + \frac{1}{k(\theta)}Kx_2]\ . \end{aligned}$$

Using the inequality

$$ab \leq \frac{1}{2\mu}a^2 + \frac{\mu}{2}b^2$$

obtain the estimates

$$\begin{aligned} Lx_2g(x,t,\theta) &\leq \frac{1}{2\mu}(Lx_2)^2 + \frac{\mu}{2}g^2(x,t,\theta) \\ Lx_2\frac{1}{k(\theta)}Kx_2 &\leq \frac{1}{2\mu}(Lx_2)^2 + \frac{\mu}{2k^2(\theta)}(Kx_2)^2\ . \end{aligned}$$

These yield

$$\dot{V}_2(x_2) \le -|x_{21}|^2 + k(\theta)\Big[Lx_2 u + \frac{1}{\mu}(Lx_2)^2 + \frac{\mu}{2}\Big(g^2(x,t,\theta) + \frac{(Kx_2)^2}{k^2(\theta)}\Big)\Big] .$$

Set now

$$u = -\Big(\frac{1}{\mu} + k_0\Big)Lx_2 = K_2 x_2 \tag{4.48}$$

where $k_0 > 0$ is any fixed number. This yields

$$\dot{V}_2(x_2) \le -|x_{21}|^2 - k_0 k(\theta)(Lx_2)^2 + \frac{\mu}{2}\Big(k(\theta)g^2(x,t,\theta) + \frac{(Kx_2)^2}{k(\theta)}\Big) ,$$

that is, bearing in mind the estimate assumed for $g^2(x,t,\theta)$,

$$\begin{aligned}\dot{V}_2(x_2) \;\; &\le \;\; -|x_{21}|^2 - k_0 k(\theta)(Lx_2)^2 \\ &\quad + \frac{\mu k(\theta)}{2}\alpha(V_1(x_1)) + \frac{\mu}{2}\Big(k(\theta)\gamma(|x_2|^2) + \frac{(Kx_2)^2}{k(\theta)}\Big) .\end{aligned}$$

By construction, the quadratic form

$$|x_{21}|^2 + k_0 k(\theta)(Lx_2)^2$$

is positive definite. Since θ ranges over a compact set, there exists a number $\lambda > 0$, which only depends on K_1, such that

$$\lambda |x_2|^2 \le |x_{21}|^2 + k_0 k(\theta)(Lx_2)^2$$

for all θ and x_2.

Recall that $\gamma(r)$ is continuously differentiable near the origin. Thus, over any compact set $\mathcal{K}$, $\gamma(|x_2|^2)$ can be estimated from above by a function of the form $\rho|x_2|^2$ where $\rho > 0$ depends of course on the set $\mathcal{K}$. Since θ ranges over a compact set as well, it is then deduced that, for any $d > 0$, there exists a number $\delta_d > 0$ such that

$$k(\theta)\gamma(|x_2|^2) + \frac{(Kx_2)^2}{k(\theta)} \le \delta_d |x_2|^2$$

for all $|x_2| < d$.

Thus,

$$\dot{V}_2(x_2) \le -\lambda|x_2|^2 + \frac{\mu\delta_d}{2}|x_2|^2 + \frac{\mu k(\theta)}{2}\alpha(V_1(x_1))$$

for all x_1 and all $|x_2| < d$. Given any $b > 0$, choose now μ such that

$$\frac{\mu\delta_d}{2} \leq \frac{\lambda}{2}, \qquad \frac{k(\theta)\mu}{2} \leq b$$

for all θ, to obtain

$$\dot{V}_2(x_2) \leq -\frac{\lambda}{2}|x_2|^2 + b\alpha(V_1(x_1)) .$$

Since $V_2(x_2) \leq \kappa|x_2|^2$ for some $\kappa > 0$, the result follows. ◁

Proposition 4.6 *Consider the system*

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + NH\eta \\ \dot{\tilde{z}} &= Z(\theta)\tilde{z} + p(H\eta, t, \theta) \\ \dot{\eta} &= F\eta + Gk(\theta)(T\tilde{\xi}_1 + M\eta + q(\tilde{z}, \eta, t, \theta)) . \end{aligned} \tag{4.49}$$

Suppose assumption A holds. For any compact set S of initial conditions $(\tilde{\xi}_1(0), \tilde{z}(0), \eta(0))$ there exist matrices T, M such that the equilibrium $(\tilde{\xi}, \tilde{z}, \eta) = (0, 0, 0)$ is asymptotically stable, with a basin of attraction which contains the set S.

Proof. Set

$$K_2 = (\,T \quad M\,)$$

and let system (4.49) be rewritten in the form

$$\begin{aligned} \dot{x}_1 &= Z(\theta)x_1 + p(HEx_2, t, \theta) \\ \dot{x}_2 &= (A + Bk(\theta)K_2)x_2 + Bk(\theta)q(x_1, Ex_2, t, \theta) . \end{aligned} \tag{4.50}$$

Given the number $c > 0$ in (4.45), let $c' > 0$ be any number such that $c|x_2|^2 \leq c'V_2(x_2)$. Then, from the previous Lemma we know that, for any choice of $b > 0$ and $d > 0$ there is K_2 such that,

$$\begin{aligned} &\frac{\partial V_1}{\partial x_1}[Z(\theta)x_1 + p(HEx_2, t, \theta)] \leq -\alpha(V_1(x_1)) + c'V_2(x_2) \\ &\frac{\partial V_2}{\partial x_2}[(A + Bk(\theta)K_2)x_2 + Bk(\theta)q(x_1, Ex_2, t, \theta)] \\ &\qquad \leq -aV_2(x_2) + b\alpha(V_1(x_1)) \end{aligned} \tag{4.51}$$

for all x_1, for all x_2 such that $|x_2| \leq d$, for all t, for all $\theta \in \Theta$.

Set

$$\chi_1(r) = \alpha^{-1}(c'r), \qquad \chi_2(r) = \frac{b}{a}\alpha(r) .$$

The two inequalities in (4.51) show that, if $V_2(x_2) > \chi_2(V_1(x_1))$ then $V_2(x_2)$ is decreasing along the trajectories of the system, while if $V_1(x_1) > \chi_1(V_2(x_2))$ then $V_1(x_1)$ is decreasing along the trajectories of the system. The corresponding regions in the (V_1, V_2) plane have a nonempty intersection if the (small gain) condition

$$\chi_1^{-1}(r) > \chi_2(r)$$

is fulfilled for all $r > 0$. This occurs, for instance, if

$$b = \frac{a}{4c'} .$$

Note now that, if b is chosen in this way, the function

$$\sigma(r) = 2\chi_2(r)$$

satisfies

$$\chi_1^{-1}(r) > \sigma(r) > \chi_2(r)$$

for all $r > 0$. Then, define

$$W(x_1, x_2) = \max\{V_2(x_2), \sigma(V_1(x_1))\} .$$

Note that this (continuous, positive definite and proper) function only depends on the function $V_1(x_1)$ introduced in the Assumption A and on the matrix K_1 chosen, once for all, in the way indicated at the beginning of the previous Lemma. In particular, $W(x_1, x_2)$ does not depend on the gain matrix K_2.

To prove stability, set

$$\begin{aligned} \Omega_c &= \{(x_1, x_2) : W(x_1, x_2) \le c\} \\ B_d &= \{(x_1, x_2) : |x_1|^2 + |x_2|^2 \le d^2\} . \end{aligned}$$

Let S be any compact set of initial conditions in the (x_1, x_2) space and choose $\bar{c}$ and d such that

$$S \subset \Omega_{\bar{c}} \subset B_d .$$

According to the value of d thus determined and to the value of b determined above, choose the matrix K_2 so as to render (4.51) fulfilled for all $(x_1, x_2) \in \Omega_{\bar{c}}$. Then, using the property that $V_2(x_2)$ is decreasing whenever $V_2(x_2) > \chi_2(V_1(x_1))$ and $V_1(x_1)$ is decreasing whenever $V_1(x_1) > \chi_1(V_2(x_2))$, it is easy to see that, for any $c \le \bar{c}$, the set Ω_c is positively invariant, and that any trajectory originating in $\Omega_{\bar{z}}$ converges to the origin as t tends to ∞.◁

Remark. Note that the matrix K_2 which renders (4.51) fulfilled has the form (4.48), namely

$$K_2 = -\Big(\frac{1}{\mu} + k_0\Big)L$$

(where L is a fixed matrix which depends on K_1 and k_0 is a fixed number). As shown in the proof of Lemma 4.5, higher values of the parameter d and lesser values of the parameter b in (4.47) require lesser values of μ. Thus, asymptotic stability of (4.49) with arbitrarily large basin of attraction is achieved via linear high-gain state-feedback. ◁

The result described in this Proposition 4.6 will be used in a moment, in the proof of the main robust stability result.

Return now to the original problem, that is the problem of robust and semiglobal regulation of the plant (4.20), and consider a control law of the form

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + Le \\ \dot{\xi}_1 &= \Phi\xi_1 + Ne \\ u &= \psi(M\xi_0) + T\xi_1 \, , \end{aligned} \tag{4.52}$$

in which $\dim(\xi_0) = n$ and $\psi(\cdot)$ is a function satisfying $\psi(a) = a$ if $|a|$ is small. This is indeed a control law of the form (4.22) and therefore the result of Proposition 4.4 apply. If the hypotheses of this Proposition hold, then the corresponding closed-loop system has a globally defined invariant manifold on which the tracking error is zero. As shown at the beginning of this section, after the change of coordinates (4.35), the problem can be viewed as the problem of finding matrices K, L, N, T, M and a function $\psi(a)$ such that the equilibrium $(\xi_0, \tilde{\xi}_1, \tilde{z}, \tilde{x}) = (0,0,0,0)$ of system (4.38), in which we set $\alpha(\xi_0) = \psi(M\xi_0)$, i.e. system

$$\begin{aligned} \dot{\xi}_0 &= K\xi_0 + Lc(\mu)\tilde{x}_1 \\ \dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + Nc(\mu)\tilde{x}_1 \\ \dot{\tilde{z}} &= Z(\mu)\tilde{z} + \tilde{p}_0(c(\mu)\tilde{x}_1, exp(St)w^\circ, \mu) \\ \dot{\tilde{x}} &= F(\mu)\tilde{x} + G(\mu)(\psi(M\xi_0) + T\tilde{\xi}_1) + \tilde{P}(\tilde{z}, \tilde{x}, exp(St)w^\circ, \mu) \, , \end{aligned} \tag{4.53}$$

is asymptotically stable, with a basin of attraction containing a fixed compact set of initial states, robustly in w° and μ (note that the change of coordinates (4.35), although depending on w°, μ and t, maps compact sets of initial conditions into compact sets of initial conditions).

The additional change of coordinates (4.41), which still maps compact sets of initial conditions into compact sets of initial conditions, brings system (4.53) to a system of the following structure

$$\begin{aligned}
\dot{\tilde{\xi}}_0 &= K\tilde{\xi}_0 + LH\eta \\
\dot{\tilde{\xi}}_1 &= \Phi\tilde{\xi}_1 + NH\eta \\
\dot{\tilde{z}} &= Z(\theta)\tilde{z} + p(H\eta, t, \theta) \\
\dot{\eta} &= F\eta + Gk(\theta)(T\tilde{\xi}_1 + \psi(M\xi_0) + q(\tilde{z}, \eta, t, \theta)) \,.
\end{aligned} \tag{4.54}$$

in which K, L, N, T, M and $\psi(a)$ are to be determined in such a way as to render the equilibrium $(\xi_0, \tilde{\xi}_1, \tilde{z}, \eta) = (0,0,0,0)$ asymptotically stable, with a basin of attraction containing a fixed compact set of initial states, robustly in θ.

We have already proven a result showing that, if ξ_0 were equal to η and $\psi(a) = a$, then matrices N, T, M could be found yielding such a robust stability property. Thus, it remains to show how K and L can be determined. To this end, we will use a method proposed by Esfandiari and Khalil, which consists in the following: choose for $\psi(\cdot)$ a saturation function, namely

$$\psi(a) = U_{\max}\text{sat}(\frac{a}{U_{\max}}) \tag{4.55}$$

where $U_{\max} > 0$ is a large number and choose for

$$\dot{\xi}_0 = K\xi_0 + Le$$

the structure of a high-speed observer, i.e. a system of the form

$$\dot{\xi}_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \xi_0 + \begin{pmatrix} Ra_{n-1} \\ R^2 a_{n-2} \\ \cdot \\ R^{n-1}a_1 \\ R^n a_0 \end{pmatrix} (e - H\xi_0) \tag{4.56}$$

in which $R > 0$ is a large number. As a matter of fact, it is possible to prove that the following result holds.

Proposition 4.7 *Consider the system*

$$\begin{array}{rcl} \dot{\tilde{\xi}}_0 & = & K\tilde{\xi}_0 + LH\eta \\ \dot{\tilde{\xi}}_1 & = & \Phi\tilde{\xi}_1 + NH\eta \\ \dot{\tilde{z}} & = & Z(\theta)\tilde{z} + p(H\eta, t, \theta) \\ \dot{\eta} & = & F\eta + Gk(\theta)(T\tilde{\xi}_1 + \psi(M\xi_0) + q(\tilde{z}, \eta, t, \theta)) \,. \end{array} \tag{4.57}$$

in which K, L have the structure specified in (4.56), $\psi(a)$ has the structure specified in (4.55), N is such that the pair (Φ, N) is controllable, $p(\eta_1, t, \theta)$ and $q(\tilde{z}, \eta, t, \theta)$ are bounded functions of t, vanishing at $(\tilde{z}, \eta) = (0, 0)$ for all t and θ, $k(\theta) > 0$ for all θ, and θ is a $(r+p)$-tuple of unknown parameters, ranging over a fixed compact set Θ. Moreover, let $a_0, a_1, \ldots, a_{n-1}$ in (4.56) be such that the matrix

$$\bar{K} = \begin{pmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

has all eigenvalues with negative real part.

Given any compact set $S \subset \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^n$, there exist matrices T and M, a number $U^* > 0$ and a number $R^* > 0$, such that, if $U_{\max} = U^*$ and $R > R^*$, every trajectory of (4.57) with initial condition in a compact set of the form $\xi_0(0) \times S$ converges to the equilibrium as t tends to ∞.

Proof. Consider the (R-dependent) change of coordinates

$$\chi = D(R)(\eta - \xi_0)$$

with

$$D(R) = \mathrm{diag}\{R^{n-1}, \ldots, R, 1\} \,,$$

and set, as before,

$$x_1 = \tilde{z}, \qquad x_2 = \begin{pmatrix} \tilde{\xi}_1 \\ \eta \end{pmatrix}, \qquad E = (0 \quad I_n) \,.$$

Finally, define

$$v = v(x_2, \chi) = \psi(MEx_2 - MD^{-1}(R)\chi) - MEx_2 \,.$$

Using these new coordinates, system (4.57) can be rewritten in the form

$$\begin{aligned}
\dot{\chi} &= R\bar{K}\chi + Gk(\theta)[K_2x_2 + q(x_1, Ex_2, t, \theta) + v(x_2, \chi)] \\
\dot{x}_1 &= Z(\theta)x_1 + p(HEx_2, t, \theta) \\
\dot{x}_2 &= (A + Bk(\theta)K_2)x_2 + Bk(\theta)q(x_1, Ex_2, t, \theta) + Bk(\theta)v(x_2, \chi)
\end{aligned} \tag{4.58}$$

where

$$A = \begin{pmatrix} \Phi & NH \\ 0 & F \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ G \end{pmatrix}$$
$$K_2 = \begin{pmatrix} T & M \end{pmatrix}$$

and F, G, H are as in (4.43). Set also $x = \mathrm{col}(x_1, x_2)$ and

$$\Omega_c = \{x : W(x) \leq c\} ,$$

where $W(x)$ is the function introduced in the proof of Proposition 4.6, namely

$$W(x) = \max\{V_2(x_2), \frac{1}{2c'}\alpha(V_1(x_1))\} .$$

Let $\bar{c}$ be such that

$$S \subset \Omega_{\bar{c}} .$$

According to the value of $\bar{c}$ thus determined choose, in view of Lemma 4.5, the matrix K_2 so as to render (4.51), in which we put $b = a/4c'$, fulfilled for all $x \in \Omega_{\bar{c}+1}$. Then, choose $U_{\max}$ as the maximum value of $|MEx_2|$ on the set $\Omega_{\bar{c}+1}$. Thus, if $x \in \Omega_{\bar{c}+1}$,

$$\psi(MEx_2) = MEx_2$$

and therefore,

$$v(x_2, 0) = 0$$

for all $x \in \Omega_{\bar{c}+1}$. Moreover, it is also easy to check that there exists a bounded positive nondecreasing continuous function $\gamma(a)$ with $\gamma(0) = 0$, which is independent of R (if $R > 1$), such that

$$|v(x_2, \chi)| \leq \gamma(|\chi|) \tag{4.59}$$

for all $(x, \chi) \in \Omega_{\bar{c}+1} \times \mathbb{R}^n$.

To prove the result, we begin by observing that the first equation in (4.58) can be rewritten in the form

$$\dot{\chi} = R\bar{K}\chi + G\phi_1(x, \chi, t, \theta) \tag{4.60}$$

where $\phi_1(x, \chi, t, \theta)$ satisfies

$$|\phi_1(x, \chi, t, \theta)| \leq \beta \tag{4.61}$$

all $(x, \chi) \in \Omega_{\bar{c}+1} \times \mathbb{R}^n$, for all t, for all $\theta \in \Theta$ and all $R > 1$, where $\beta > 0$ is a fixed number.

As a consequence, it is possible to show that for any $t_1 > 0$ and any ϵ there exists a number $R^* > 0$ such that, if $R > R^*$, for any initial condition in $\xi_0(0) \times \mathcal{S}$

$$x(t) \in \Omega_{\bar{c}+1} \text{ for all } t \in [0, T] \quad \Rightarrow \quad |\chi(t_1)| \leq \epsilon, \text{ for all } t \in [t_1, T] , \tag{4.62}$$

where $T \geq t_1$ is any number. To this end, let P be a positive definite solution of $P\bar{K} + \bar{K}^{\mathrm{T}}P = -I$ and observe that, if $x(t) \in \Omega_{\bar{c}+1}$ for all t in some interval $[0, T]$, the function $Q(\chi) = \chi^{\mathrm{T}}P\chi$ satisfies (see (4.60) and (4.61))

$$\begin{aligned} \frac{\partial Q}{\partial \chi}\dot{\chi} &\leq -R|\chi|^2 + 2|\chi^{\mathrm{T}}P|\beta \leq -(R - \frac{k_1}{\mu})|\chi|^2 + \mu\beta^2 \\ &\leq -(R - \frac{k_1}{\mu})k_2 Q(\chi) + \mu\beta^2 \leq -aQ(\chi) + \mu\beta^2 \end{aligned}$$

where $k_1 > 0$ and $k_2 > 0$ are numbers depending only on P, μ is any positive number, and

$$a = (R - \frac{k_1}{\mu})k_2 .$$

By standard comparison arguments, it is deduced that

$$|\chi(t)|^2 \leq k_3\Big[e^{-at}|\chi(0)|^2 + \frac{1 - e^{-at}}{a}\mu\beta^2\Big] ,$$

where $k_3 > 0$ is a number depending only on P. Fix ϵ and choose μ to satisfy $2k_3\mu\beta^2 \leq \epsilon^2$, so that, if $a > 1$,

$$|\chi(t)|^2 \leq k_3 e^{-at}|\chi(0)|^2 + \frac{\epsilon^2}{2} .$$

Fix $t_1 > 0$ and observe that

$$\lim_{R\to\infty} e^{-at_1}|\chi(0)|^2 = 0 ,$$

because $|\chi(0)| = |D(R)(\eta(0) - \xi_0(0))|$ is bounded by a polynomial of order $n-1$ in R. In particular, since $\xi_0(0)$ is fixed and $\eta(0)$ ranges over a compact set, there is R^*, independent of $\eta(0)$, such that

$$k_3 e^{-at_1} |\chi(0)|^2 \leq \frac{\epsilon^2}{2}$$

for every $R > R^*$ and every $\eta(0)$ and the result (4.62) follows.

Now, consider the second and third equations of (4.58), namely

$$\begin{aligned} \dot{x}_1 &= Z(\theta)x_1 + p(HEx_2, t, \theta) \\ \dot{x}_2 &= (A + Bk(\theta)K_2)x_2 + Bk(\theta)q(x_1, Ex_2, t, \theta) + Bk(\theta)v(x_2, \chi), \end{aligned} \tag{4.63}$$

which differ from the equations characterizing system (4.50) only by the additional term $Bk(\theta)v(x_2, \chi)$ which affects the second one.

From the arguments previously used in the proof of Lemma 4.5 and from the estimate (4.59), we know that the functions $V_1(x_1)$ and $V_2(x_2)$ satisfy

$$\begin{aligned} &\frac{\partial V_1}{\partial x_1}[Z(\theta)x_1 + p(HEx_2, t, \theta)] \leq -\alpha(V_1(x_1)) + c'V_2(x_2) \\ &\frac{\partial V_2}{\partial x_2}[(A + Bk(\theta)K_2)x_2 + Bk(\theta)q(x_1, Ex_2, t, \theta)] \\ &\qquad \leq a[-V_2(x_2) + \frac{1}{4c'}\alpha(V_1(x_1)) + k\gamma(|\chi|)] , \end{aligned} \tag{4.64}$$

for all $(x, \chi) \in \Omega_{\bar{c}+1} \times \mathbb{R}^n$, for all t, for all $\theta \in \Theta$ and all $R > 1$, where $k > 0$ is some number depending only on $V_2(x_2)$. Thus, $V_2(x_2(t))$ is decreasing so long as

$$V_2(x_2(t)) > \frac{1}{4c'}\alpha(V_1(x_1(t))) + k\gamma(|\chi(t)|) .$$

Recall also that the two inequalities (4.51) have enabled us to show that, for all $x(0) \in S$, the function $W(x(t))$ is decreasing along the trajectories of (4.50). Similar arguments enable us now to show that the trajectories of the "perturbed" system (4.63) converge to a ball of arbitrarily small radius.

To this end, take any $x \in S$. Let $\delta > 0$ be any (small) number, satisfying $\delta < |x(0)|$ and such that

$$B_\delta = \{x : |x| \leq \delta\} \subset \text{int}(\Omega_{\bar{c}+1}) .$$

Then, choose ϵ so that

$$\{(x_1, x_2) : V_1(x_1) \le \alpha^{-1}(4c'k\gamma(\epsilon)), V_2(x_2) \le 2k\gamma(\epsilon)\} \subset B_\delta \ ,$$

and consider the set

$$\Gamma = \{x \in \Omega_{\bar{c}+1} : |x| \ge \delta\} \ .$$

Clearly, $x(0)$ is in the interior of Γ and, if t_1 is sufficiently small, so is also $x(t)$ for all $t \in [0, t_1]$. Thus, $|\chi(t_1)| \le \epsilon$. The previous arguments prove that, for $t \ge t_1$, $x(t)$ remains in the interior of $\Omega_{\bar{c}+1}$. In fact, so long as $x(t) \in \Gamma$, (4.62) implies $\chi(t) \le \epsilon$ and the arguments already used in the proof of Proposition 4.6 show that $W(x(t))$ is decreasing until $x(t)$ enters the region

$$\{(x_1, x_2) : V_1(x_1) \le \alpha^{-1}(4c'k\gamma(\epsilon)), V_2(x_2) \le 2k\gamma(\epsilon)\} \ .$$

Thus $x(t)$ cannot reach the boundary of $\Omega_{\bar{c}+1}$, because there $W(x) > W(x(0))$. On the contrary, in finite time $x(t)$ leaves Γ through the boundary of B_δ.

Having proven that, in finite time, both $x(t)$ and $\chi(t)$ enter a ball of arbitrarily small radius centered at the equilibrium $(x, \xi) = (0, 0)$, the proof can be completed by showing that the latter is locally asymptotically (actually, exponentially) stable. To this end, observe that if, $|x|$ and $|\chi|$ are sufficiently small (and $R > 1$),

$$|M(Ex_2 - D^{-1}(R)\chi)| < U_{\max}$$

and therefore

$$v(x_2, \chi) = -MD^{-1}(R)\chi \ .$$

Thus, the problem is to establish local asymptotic stability of the system

$$\begin{aligned} \dot{\chi} &= R\bar{K} + Gk(\theta)[K_2x_2 + q(x_1, Ex_2, t, \theta) - MD^{-1}(R)\chi] \\ \dot{x} &= \phi_2(x, t, \theta) - \delta(\theta)MD^{-1}(R)\chi \ . \end{aligned} \tag{4.65}$$

Recall assumption A and observe that the estimate (4.46), since $\alpha(r)$ and $\gamma(r)$ are continuously differentiable near $r = 0$ and $V_1(x_1)$ is a smooth function, implies that, for small $|x|$,

$$|q(x_1, Ex_2, t, \theta)|^2 \le d|x|^2$$

where $d > 0$ is a fixed number. Thus, arguments identical to those used above show that for any $\mu > 0$ there is a R^* such that, for all $R > R^*$, the quadratic form $Q(\chi)$ satisfies an inequality of the form

$$\frac{\partial Q}{\partial \chi}\dot{\chi} \leq -a_0|\chi|^2 + \mu|x|^2 \tag{4.66}$$

where a_0 is a positive fixed number.

Assumption A implies that the equilibrium $x_1 = 0$ of system

$$\dot{x}_1 = Z(\theta)x_1$$

is globally asymptotically stable, actually exponentially stable (because the system is linear). Thus, for each θ, there exists a matrix $S(\theta) > 0$ such that the quadratic form $\tilde{V}_1(x_1) = x_1^T S(\theta)x_1$ satisfies

$$\frac{\partial \tilde{V}_1}{\partial x_1} Z(\theta)x_1 = -|x_1|^2 \,.$$

As a consequence, the derivative of $\tilde{V}_1(x_1)$ along the trajectories of (4.65), for small $|x|$, satisfies

$$\frac{\partial \tilde{V}_1}{\partial x_1}\dot{x}_1 \leq -a_1|x_1|^2 + b_1|x_2|^2 \,, \tag{4.67}$$

where a_1 and b_1 are positive fixed numbers.

Finally, arguments identical to those used in the proof of Lemma 4.5 show that the quadratic form $V_2(x_2)$ there introduced satisfies, for small $|x|$,

$$\frac{\partial V_2}{\partial x_2}\dot{x}_2 \leq -a_2|x_2|^2 + b_2|x_1|^2 + c|\chi|^2 \,, \tag{4.68}$$

where a_2, b_2, c are positive fixed numbers and b_2 can be rendered arbitrarily small (by proper choice of K_2).

From these properties, the required result follows from a repeated application of the following "small-gain" argument, whose derivation is straightforward and hence omitted. Suppose $U_1(z_1)$ and $U_2(z_1)$ are positive definite quadratic forms satisfying

$$\begin{aligned}
\frac{\partial U_1(z_1)}{\partial z_1} f_1(z_1, z_2) &\leq -a_1|z_1|^2 + b_1|z_2|^2 \\
\frac{\partial U_2(z_2)}{\partial z_2} f_2(z_1, z_2, u) &\leq -a_2|z_2|^2 + b_2|z_1|^2 + c|u|^2 \,,
\end{aligned}$$

for small $|z_1|, |z_2|, |u|$, where a_1, a_2, b_1, b_2, c are positive fixed numbers. Then, if

$$\frac{b_1 b_2}{a_1 a_2} < 1 ,$$

there is $\alpha > 0$ such that the positive definite quadratic form

$$W(z_1, z_2) = \frac{1}{a_1} U_1(z_1) + \frac{\alpha}{a_2} U_2(z_1)$$

satisfies

$$\begin{pmatrix} \frac{\partial W}{\partial z_1} & \frac{\partial W}{\partial z_2} \end{pmatrix} \begin{pmatrix} f_1(z_1, z_2) \\ f_2(z_1, z_2, u) \end{pmatrix} \leq -a|z_1|^2 - a|z_2|^2 + b|u|^2$$

for small $|z_1|, |z_2|, |u|$, where a and b are positive fixed numbers. ◁

Bibliographical Notes

The problem of output regulation for multivariable linear system has been studied by many authors, among whom we mention, for instance, Smith and Davison [44], Davison [7], Francis and Wonham [16], Francis [14], Wonham [50]. In particular, the work [14] of Francis has shown that the solvability of a multivariable linear regulator problem corresponds to the solvability of a system of two linear matrix equations and this is in turn equivalent, as illustrated by Hautus in [17], to a certain property of the transmission zeros of a composite system which incorporates the plant and the exosystem. As observed before, the *regulator equations* (2.10) are the nonlinear counterpart for the above-mentioned pair of linear equations found in the work of Francis; likewise, conditions for the existence of solutions of the regulator equations (2.10) discussed Chapter 3 are the nonlinear counterpart for the properties of transmission zeros identified by Hautus. The work of Francis and Wonham [16] has shown that, in the case of error feedback, any regulator which solves the problem in question incorporates a model of the dynamical system generating the reference and/or the disturbance signals which must be tracked and/or rejected. This property is commonly known as the *internal model principle*.

More recently, some authors have considered the problem of output regulation also for nonlinear systems. The work of Hepburn and Wonham [19] presents a rather complete extension of the notion of internal model and its properties in the context of problems of output regulation for nonlinear systems defined on differentiable manifolds. The work of Anantharam and Desoer [1] investigates conditions for the existence of regulators for the purpose of tracking constant reference signals. The work of Di Benedetto [10] describes conditions for the existence of regulators in the case of one-dimensional exosystem. The work of Huang and Rugh [27] presents a complete extension of the conditions established by Francis to the case in which the exosystem generates constant reference signals. The work of Isidori and Byrnes [34] has shown how the results established by Francis can be extended to the general case of a nonlinear plant and a nonlinear exosystem generating time-varying reference signals and/or disturbances, and how the interpretation established by Hautus finds its natural nonlinear extension in terms of the concept of zero dynamics. The approach of [34] to the problem of output regulation was further pursued by Huang and Rugh in [28], [29]. In particular, the first paper improved the stabilization results of [34] (by weakening the requirement of stability in the first approximation) and addressed the issue of finding a power series expansion of the solution $\pi(w)$ of the regulator equation (2.10). The second paper elaborated on the property that if the solution in question is determined only up to a certain degree of accuracy, then output regulation can be secured up to a steady-state error of the same degree. An exhaustive presentation of a number of issues related to polynomial approximation and/or power series expansions for the determination of the solution of (2.10) can also be found, together with several of important related results, in the paper [39] by Krener. The concept of internal model and its role in the construction of a controller solving the problem of output regulation was further refined by Isidori in [30], where it is shown that the notion of internal model can be best expressed in terms of the concept of *immersion* of a system into another system, introduced earlier by Fliess in the context of problems of system realization and system equivalence (see [12], [13]). Based on this interpretation of the concept of internal model, [30] presented (for the first time, to the best of our knowledge) a complete set of necessary and sufficient conditions for the existence of a solution for the problem of local output regulation for a nonlinear system.

The problem of output regulation for *uncertain* multivariable linear systems has been addressed and solved by Davison [7] and Francis [14]. In the case of nonlinear systems the problem in question was originally addressed in [16] and [20], where it was shown that, when the exogenous input is *constant* (i.e. set point control under constant disturbances), the incorporation of an internal model into the compensator (i.e. integral control) suffices to guarantee output regulation in the presence of small parameter variations (i.e. a ***structurally stable design***, in the terminology of [14], [20], is achieved). However, Byrnes and Isidori [4] showed that, in the case of time-varying inputs, a linear controller which is robust with respect to parameter variations affecting the linear approximation may no longer be robust if the parameter uncertainties affect some nonlinear term.

A breakthrough in the direction of solving the problem of dealing with parameter uncertainties in a problem of nonlinear regulation was the crucial observation by Khalil [36] (see also [37]) that, in the presence of nonlinearities with unknown parameters, the internal model must not only be able to generate inputs corresponding to the trajectories of the exosystem, but also a number of their "higher order" nonlinear "deformations". For example, in the case of a cubic nonlinearity with unknown coefficient and sinusoidal reference output, the internal model must generate the sinusoid in question *and* its third harmonic. This key idea was also independently elaborated by Huang and Lin, in [24], [25], [26] and by Delli Priscoli in [8], [9]. In particular Huang and Lin, appealing to concept of "regulation of order k" (namely, regulation up to a steady-state error which is infinitesimal of order

k with respect to the amplitude of the disturbance input) introduced earlier in [29], provided in [24] a methodology for the design of a controller which, regardless of small parameter perturbations, achieves regulation of order k. This methodology was proven in [25] to yield exact regulation, regardless of small parameter variations, for some relevant classes of nonlinear systems [25]. In fact, [25] shows that if the function $c(w)$, which renders the regulator equations (2.10) satisfied for some $\pi(w)$, is a *polynomial* of (some) degree k in w and the exosystem is a *linear system* then it is possible to obtain *structurally stable* regulation by designing an internal model which generates all the exogenous inputs as well all higher harmonics, up to order k. Delli Priscoli, in [8] and [9], arrived at the (more general) conclusion that structurally stable regulation is possible if the family of all functions of the form $u = c(w(t))$, where $w(t)$ is any exogenous input generated by the exosystem, can be seen as a subset of the set of all possible solutions of a fixed ordinary differential equation. This sufficient condition was later proven to be also necessary in [30], where a complete set of necessary and sufficient conditions for the existence of a solution for the problem of structurally stable local output regulation for a nonlinear system was presented.

The problem of *robust regulation*, i.e. regulation in the presence of parameter uncertainties ranging over *pre-assigned* compact *sets*, has been studied by Khalil [37], by Byrnes, Delli Priscoli, Isidori and Kang [3], by Mahmoud and Khalil [41], [42] and by Isidori [32] (see also [33]). Of these papers, [3] considers the case of robust *local* output regulation, and presents the method described here in section 5.2. The other papers consider the problem of robust *semiglobal* regulation, i.e. the problem of designing of a controller yielding asymptotic regulation for any initial condition over an arbitrarily large (but fixed) compact set, robustly with respect to unknown parameters also ranging over an arbitrarily large (but fixed) compact set. In particular, paper [37] studies a class of system having relative degree equal to the dimension of the state space (i.e. having a trivial zero dynamics) and utilizes a technique developed earlier by Esfandjari and Khalil in [11] to design an error-feedback controller in which the components of the internal state are estimated by means of a "saturated" high-speed observer. Paper [41] provides a set of sufficient conditions under which the approach of [37] can be extended to the case of systems having a nontrivial zero dynamics. Finally, paper [32] provides an extension of the results of [37], [41] and establishes a significant bridge between the results of these papers and the general (but local) approach of [30]. In particular, paper [32] presents the main semiglobal regulation result described here in section 4.5, whose proof requires a blend of a number of most important robust stabilization result recently presented in the literature. Once the existence of a globally defined invariant manifold on which the error is zero has been established (using techniques presented in [38]), the problem of robust regulation becomes similar to the problem of semiglobal stabilization via output feedback addressed and solved by Teel and Praly in [48] and [49]. The stability proof described here, which follows a pattern similar to a proof presented in [37], uses semiglobals robust stablilization methods of Krstic, Kanellakopoulos, Kokotovic [40], the implications of the concept of input-to state stability of Sontag [45], the idea of a saturated high-speed observer of Esfandjari and Khalil [11] (already used earlier, to the same purpose, in [37]), and a small-gain stability theorem of Hill and Moylan [22].

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